

EQUIVARIANT ZARISKI STRUCTURES

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ABSTRACT. A category of equivariant algebras is defined, after introducing some important examples. To each equivariant algebra, a first order theory is assigned. Model theoretic results are established (uncountable categoricity, quantifier elimination to the level of existential formulas) and that an appropriate dimension theory exists for models, making them Zariski structures. A functor from the category of equivariant algebras to the category of Zariski structures is defined and further properties of equivariant structures are briefly discussed.

1. INTRODUCTION

The present paper is comprised of a geometric model-theorist's attempts to do noncommutative algebraic geometry. The latter endeavour takes as its starting point the possibility of extending the anti-equivalence of categories \mathbf{CRing} and \mathbf{AffSch} (which denote the categories of commutative unital rings and affine schemes respectively) to arbitrary rings and putative 'noncommutative schemes'; namely establishing that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{CRing}^{op} & \rightleftarrows & \mathbf{AffSch} \\ \downarrow & & \downarrow \\ \mathbf{Ring}^{op} & \rightleftarrows & \mathbf{NSch} \end{array}$$

where \mathbf{NSch} is a candidate for the category of noncommutative schemes. The author has adopted the viewpoint that geometric model theory can only interact with noncommutative algebraic geometry via the theory of Zariski structures; and indeed does so most naturally. This is a viewpoint he now wishes to justify.

Firstly, an investigation of important applications of geometric model theory to questions of *commutative* algebraic geometry (specifically diophantine geometry) indicates that a difference of approach is currently needed if our endeavours are to succeed. The methodology of such applications can be summarized as follows. One selects appropriate structures (e.g. algebraically closed fields, differentially closed fields, separably closed fields), establishes what 'stability class' the structures belong to and deduces results about definable sets by applying the relevant abstract model-theoretic tools associated with this stability class¹. Crucially, the language and techniques of geometric model theory, with its emphasis on stability and appropriate generalizations of this notion, independence and ranks, working in a universal domain etc, is closer in spirit and language to Weil's foundations than scheme theory. One certainly does not work at the level of generality of an arbitrary commutative ring. For any hope of applying model theory to noncommutative algebraic geometry, it seems that one should remain (at the very least) in a suitably geometric setting and look for geometric counterparts to suitable classes of noncommutative k -algebras, where k is an algebraically closed field. But at the same time, there seems to be no reason for suspecting that there is a nice structure whose definable subsets can be regarded as coordinate rings of a sufficiently interesting and large class of noncommutative k -algebras. It is not possible to do any 'naive' noncommutative algebraic geometry in the manner that one can work with varieties as subsets of affine or projective space. The language of schemes and category-theoretic generalizations of it are indispensable for most of the popular existing approaches to noncommutative algebraic geometry ([Mah06], [Ros95]).

¹The articles [Mac03] and [Hru98] contain an introduction to methods of geometric model theory; [B99] discusses the specific application of these methods to Mordell-Lang.

Rather, we are forced to

- find a systematic means of associating a structure to a given noncommutative k -algebra, suitably axiomatized in an appropriate language.
- ask whether these structures share any common geometry.

An association of structures to algebras should be functorial if it is to be systematic; thus we must work with a geometric category of structures not necessarily all defined in the same language. If we are aiming for an extension of commutative algebraic geometry then a basic intersection theory resembling the commutative case should exist, i.e. some well-behaved notion of dimension should exist for a large class of subsets of each noncommutative structure. It transpires that these rather basic requirements lead us to stipulate that the associated structures are Zariski structures. We work with the definition of [Zil10]:

Definition 1.1. *Let \mathbf{X} be a set. A **Zariski structure**² on \mathbf{X} consists of a Noetherian topology on \mathbf{X}^n for every $n > 0$ and an \mathbb{N} -valued dimension function \dim on non-empty projective subsets (finite unions of projections of closed subsets) satisfying the following properties:*

- (1) *The dimension of a point is 0.*
- (2) *$\dim(\mathbf{P}_1 \cup \mathbf{P}_2) = \max\{\dim \mathbf{P}_1, \dim \mathbf{P}_2\}$ for all projective subsets $\mathbf{P}_1, \mathbf{P}_2$.*
- (3) *For \mathbf{C} closed and irreducible in \mathbf{X}^n and \mathbf{C}_1 a closed subset of \mathbf{C} , if $\mathbf{C}_1 \neq \mathbf{C}$ then $\dim \mathbf{C}_1 < \dim \mathbf{C}$.*
- (4) *For \mathbf{C} irreducible and closed in \mathbf{X}^n , if $\pi : \mathbf{X}^n \rightarrow \mathbf{X}^m$ is a projection then*

$$\dim \mathbf{C} = \dim \pi(\mathbf{C}) + \min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C})$$

- (5) *For any irreducible closed \mathbf{C} in \mathbf{X}^n and projection map $\pi : \mathbf{X}^n \rightarrow \mathbf{X}^m$, there is a subset \mathbf{V} relatively open in $\pi(\mathbf{C})$ such that*

$$\min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C}) = \dim(\pi^{-1}(v) \cap \mathbf{C})$$

for every $v \in \mathbf{V} \cap \pi(\mathbf{C})$.

Moreover, projections must be semi-proper, i.e. for any closed irreducible subset \mathbf{C} of \mathbf{X}^n and projection map $\pi : \mathbf{X}^n \rightarrow \mathbf{X}^m$, there is a proper closed subset \mathbf{D} of $\overline{\pi\mathbf{C}}$ such that $\overline{\pi\mathbf{C}} \setminus \mathbf{D} \subseteq \pi\mathbf{C}$. A Zariski structure is said to be **presmooth** if for any closed irreducible subsets $\mathbf{C}_1, \mathbf{C}_2$ of \mathbf{X}^n the dimension of any irreducible component of $\mathbf{C}_1 \cap \mathbf{C}_2$ is greater than or equal to

$$\dim \mathbf{C}_1 + \dim \mathbf{C}_2 - \dim \mathbf{X}^n$$

A natural candidate for a morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ of Zariski structures is a function inducing a continuous map on \mathbf{X}^n for every n . Thus we have a category of Zariski structures with these morphisms, which we denote by **Zar**. Some familiarity with algebraic geometry (in particular results on the dimensions of fibers, [Har77], II, Exercise 3.22) will allow one to conclude that varieties are Zariski structures, and are presmooth if the varieties are smooth. Hence the category of algebraic varieties is a subcategory of **Zar**. Moreover, like schemes, Zariski structures have the advantage of being abstractly given and not as sitting in some ambient structure.

However, Zariski structures were not introduced to fulfill the purpose of being a model-theorists' answer to algebraic manifolds. Rather, they first appeared in [HZ96] as a response to the failure of Zilber's trichotomy conjecture. Roughly speaking, the trichotomy conjecture proposed that the geometry of certain subsets of models (the so-called strongly minimal sets) fell into three mutually exclusive classes; such geometries were either trivial, linear, or that of an algebraically closed field. After the ingenious refutation of this conjecture by Hrushovski, it was natural to ask whether there was a natural class of structures for which the trichotomy conjecture did hold. One-dimensional

²Technically, according to the terminology of [Zil10] we shall be defining Noetherian Zariski structures as opposed to analytic Zariski structures. Because we do not deal with the latter, for the purposes of this thesis the adjective 'Noetherian' can be dropped.

Zariski structures³ turned out to be such a class. For our purposes, two aspects of the work in [HZ96] are particularly important. Firstly, as already mentioned, projective algebraic curves are Zariski structures. Secondly, there are one-dimensional Zariski structures which are demonstrably not projective curves but are certain finite covers of them. These structures, rather than turning out to be mathematical pathologies, can be taken to be geometric objects corresponding to certain noncommutative algebras. In this regard, we mention the paper [ZS09] as providing an example of such a one-dimensional Zariski structure corresponding to a physically important algebra, namely the Heisenberg algebra. In short, Zariski structures corresponding to noncommutative algebras do exist that can be distinguished from projective curves by their geometry not being reducible to them.

Given that there are one-dimensional **non-classical** Zariski structures (those not arising from algebraic curves) and that these correspond to certain noncommutative algebras, it is natural to expect that there are higher-dimensional Zariski structures corresponding to other noncommutative algebras. The paper [Zil06] establishes exactly this: that non-classical Zariski structures can be associated to a class of noncommutative algebras, described in the paper as ‘quantum algebras at roots of unity’. The definition of such algebras can be simplified with some knowledge of ring theory and the results of [Zil06] shall be discussed in due course. The results of [ZS09] and [Zil06] provide sufficient evidence to propose the following conjecture.

Conjecture 1.1. *Let k be an algebraically closed field. Then there is a commutative diagram of functors*

$$\begin{array}{ccc}
 (\mathbf{CAlg}(k)_{fg,int})^{op} & \longrightarrow & \mathbf{Zar}^c \\
 \downarrow & & \downarrow \\
 \mathbf{Alg}(k)^{op} & \longrightarrow & \mathbf{Zar}
 \end{array}$$

$\mathbf{CAlg}(k)_{fg,int}$ $\mathbf{Alg}(k)$ \mathbf{Zar}^c	<i>Finitely generated, commutative k-algebras that are domains</i> <i>k-algebras</i> <i>Classical Zariski structures</i>	
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where the functor $\mathbf{Alg}(k)^{op} \rightarrow \mathbf{Zar}$ is an equivalence of categories.

The conjectural functor is, of course, $\mathbf{Alg}(k)^{op} \rightarrow \mathbf{Zar}$ and the work in this paper has the construction of this functor as a focal point. To date, a general means of constructing a suitable such functor has not been found. As far as the author’s work is concerned, the most fruitful *modus operandi* (both conceptually and pragmatically) has been the following:

- (1) Rather than attempting to construct a general functor $\mathbf{Alg}(k)^{op} \rightarrow \mathbf{Zar}$, isolate an interesting subcategory of k -algebras \mathbf{A} that contains a suitably large subcategory \mathbf{B} of the category of affine commutative k -algebras that are domains.
- (2) Constrain the algebraic characterisation of \mathbf{A} by those additional assumptions necessary to associate an \mathcal{L}_A -structure $\mathbf{nSpec} A$ to every object A of \mathbf{A} (where the language \mathcal{L}_A depends on the object A). The structure $\mathbf{nSpec} A$ should be a moduli space for certain representations of A , preferably those A -modules that ‘generate’ an interesting subcategory of the category of all left A -modules, ${}_A\mathbf{Mod}$.
- (3) Carry out an analysis of the definable subsets of $\mathbf{nSpec} A$ and conclude that $\mathbf{nSpec} A$ is a Zariski structure.

³The definition of Zariski structures appearing in [HZ96] is less general than Definition 1.1 because it stipulates that the underlying set \mathbf{X} is one-dimensional in a suitable model-theoretic sense. When \mathbf{X} is one-dimensional, both definitions coincide. We shall be dealing with Zariski structures where \mathbf{X} has dimension > 1 . Such Zariski structures will be referred to as higher-dimensional.

- (4) Extend the correspondence $A \mapsto \text{nSpec } A$ to a functor $\text{nSpec} : \mathbf{A}^{op} \rightarrow \mathbf{Zar}$ and verify that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B}^{op} & \longrightarrow & \mathbf{Zar}^c \\ \downarrow & & \downarrow \\ \mathbf{A}^{op} & \xrightarrow{\text{nSpec}} & \mathbf{Zar} \end{array}$$

- (5) Finally analyze the relationship between $\text{nSpec } A$ and ${}_A\mathbf{Mod}$ for every object A of \mathbf{A} .

It is appropriate to be a little bit more specific about syntax and related issues at this juncture. Let \mathbf{A}' be our category of k -algebras obtained after appropriate constraints are introduced in 2. Then to each algebra A in \mathbf{A}' we associate an \mathcal{L}_A -theory T_A that is first-order axiomatizable. The structure $\text{nSpec } A$ is then taken to be a large saturated model (universal domain) of T_A . In much the same way that the language of rings naturally axiomatizes the theory of algebraically closed fields, the language \mathcal{L}_A is chosen to be natural for T_A . Moreover, the Zariski structure obtained on $\text{nSpec } A$ should respect the theory T_A , in the sense that it arises from a suitable quantifier-elimination result. This particular methodology is uniquely model-theoretic and results in a topology that is rather descriptive. Crucial to this is the insistence in 2 on the structure $\text{nSpec } A$ being a moduli space for a class of A -modules. Thus $\text{nSpec } A$ incorporates the internal structure of the modules explicitly into the geometry.

We now summarize the contents of this paper. We deal with a class of algebras which are described as **equivariant**. The choice of terminology here is motivated by important structures appearing in geometric representation theory; namely those line bundles L over a variety V endowed with an action of an algebraic group G , such that

$$\text{for all } g \in G, g(L_x) = L_{gx} \text{ and } g : L_x \rightarrow L_{gx} \text{ is a linear isomorphism}$$

where L_x denotes the fiber of L at $x \in V$. Such line bundles are said to be **G -equivariant** (see [RTT07]). The structure corresponding to the Heisenberg algebra introduced in [ZS09] looked, at least superficially, to be an equivariant line bundle. However, further examination revealed some crucial differences. Firstly, there was no claim on local triviality. Secondly, whereas certain operators (\mathbf{a} and \mathbf{a}^\dagger for those familiar with the paper) did move between fibers in a manner that introduced an action of a group on the base, these two operators themselves didn't generate a group because they were not mutually inverse. It is the author's contention (and no doubt that of B. Zilber also) that such phenomena are characteristic of 'quantum' objects. Additional examples worked out in a similar vein (the quantum 2-torus by Zilber, $U_q(\mathfrak{sl}_2(k))$ for generic q by the author) suggested that an appropriate formalism could be found that treated all of these examples (and more) collectively. We discuss these examples in Section 2 and the category of equivariant algebras (denoted $\mathbf{Equiv}(k)$) is defined in Section 3. It is not a full subcategory of $\mathbf{Alg}(k)$ and an appropriate notion of a morphism in this category is given. We also show that given an object A of $\mathbf{Equiv}(k)$, we can associate a first-order \mathcal{L}_A -theory T_A to A .

Sections 4 and 5 are devoted to the model theory of T_A under an additional technical assumption on T_A (Γ -rigidity), which the key examples mentioned in Section 1 are shown to satisfy. Uncountable categoricity and quantifier elimination results are established thus leading to the expected consequences for the category of definable subsets; namely that every definable subset is constructible for an appropriate topology on models. With this topology, an appropriate dimension theory turns each model into a Zariski structure. The method of technical analysis is that of [Zil06]. It is worth remarking that the condition of Γ -rigidity encapsulates precisely what is required for T_A to possess a rich structure theory, i.e. it is only for Γ -rigid T_A that models are Zariski structures.

The final section concludes our excursion into equivariant algebras and their associated Zariski structures with the expected construction of a functor nSpec . Appendices are provided summarizing relevant background material for Lie algebras and Hopf algebras. To the author's knowledge, the structures $\text{nSpec } A$ for general equivariant A have no precedent. They are also unusual in being

able to assign to certain noncommutative algebras parametrized at a generic parameter a bone fide topological space, in contrast to the approaches to noncommutative algebraic geometry surveyed.

2. SOME EXAMPLES

Three examples of noncommutative algebras are discussed, occupying a central place in physics, the theory of quantum groups and noncommutative geometry respectively.

2.1. Weyl Algebra. Recall that for a commutative ring R , the n -th **Weyl algebra** $A_n(R)$ (for $n > 0$) is defined to be

$$R\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / I$$

where I is the ideal generated by

$$\partial_i x_j - x_j \partial_i - \delta_{ij} \quad x_i x_j - x_j x_i \quad \partial_i \partial_j - \partial_j \partial_i \quad \text{for } 1 \leq i, j \leq n$$

We shall concentrate on the first Weyl algebra $A_1(k)$ for k an algebraically closed field of characteristic 0. Firstly, we note that $A_1(k)$ can be redefined in terms of three operators $H, \mathbf{a}, \mathbf{a}^\dagger$:

$$\mathbf{H} = \frac{1}{2}(x^2 - \partial_x^2) \quad \mathbf{a} = \frac{1}{\sqrt{2}}(x + \partial_x) \quad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x)$$

Because we are working in an arbitrary algebraically closed field, $\sqrt{2}$ represents an element that squares to 2. The operator \mathbf{a}^\dagger is a formal adjoint to \mathbf{a} , as an element of the differential ring $A_1(k)$.

Proposition 2.1. *The following relations hold between $H, \mathbf{a}, \mathbf{a}^\dagger$:*

- (1) $\mathbf{a}^\dagger \mathbf{a} = \mathbf{H} - 1/2, \mathbf{a} \mathbf{a}^\dagger = \mathbf{H} + 1/2$.
- (2) $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{a} \mathbf{a}^\dagger - \mathbf{a}^\dagger \mathbf{a} = 1$.
- (3) *Putting $\mathbf{N} = \mathbf{H} - 1/2$, we have $[\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger$ and $[\mathbf{N}, \mathbf{a}] = -\mathbf{a}$. Thus we also have*

$$[\mathbf{H}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger \quad [\mathbf{H}, \mathbf{a}] = -\mathbf{a}$$

Proof. 1 and 2 are easy verification. For 3, use the fact that for any three operators A, B, C , we have the relation $[A, BC] = [A, B]C + B[A, C]$. \square

In [ZS09], a Zariski structure was associated to the Heisenberg algebra $k\langle \mathbf{P}, \mathbf{Q} \rangle / I$ where I is generated by $[\mathbf{P}, \mathbf{Q}] + i$. Similarly this algebra was also re-expressed in terms of operators $\mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger$ defined slightly differently to the above, namely by

$$\mathbf{H} = \frac{1}{2}(\mathbf{P}^2 + \mathbf{Q}^2) \quad \mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{P} - i\mathbf{Q}) \quad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathbf{P} + i\mathbf{Q})$$

The relations satisfied by these are, however, the same as in Proposition 2.1. The structure we define below is different to that of [ZS09]; indeed the following structure originally appeared as a quotient of an initial (also Zariski) structure in that paper. The latter was important insofar as it provided another example of a one-dimensional Zariski geometry (a finite cover of the projective line) not definable in an algebraically closed field. For our purposes, we can start directly with the quotient.

Definition 2.1. *We consider a two-sorted language $\mathcal{L}_{A_1} = (k, L, \pi, \mathbf{E}, \mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger)$ where*

- (1) $\pi : L \rightarrow k$ and $\mathbf{H}, \mathbf{a}, \mathbf{a}^\dagger : L \rightarrow L$ are maps.
- (2) $\mathbf{E} \subseteq L \times k$ is a relation.
- (3) *The sort k has the language of rings. The sort L comes equipped with*
 - *a map $+$: $L \times L \rightarrow L$ which is interpreted as addition of elements in each $\pi^{-1}(x)$ for $x \in k$*
 - *a map \cdot : $k \times L \rightarrow L$ which is interpreted as scalar multiplication in each fiber $\pi^{-1}(x)$.*

The \mathcal{L}_{A_1} -theory T_{A_1} says the following:

- (1) *k is an algebraically closed field of characteristic 0.*
- (2) *$\pi : L \rightarrow k$ is a surjective map; each fiber $\pi^{-1}(x)$ for $x \in k$ is a one-dimensional k -vector space.*
- (3) *For each $x \in k$, the subset $\mathbf{E}(L, x)$ is non-empty and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.*

- (4) We fix an $l \in \mathbb{Z}$, $l > 0$, l even. If $\Gamma[l]$ denotes the group of l -th roots of unity of k , then there is a free and transitive action of $\Gamma[l]$ on $\mathbf{E}(L, x)$ induced by the vector space action on the fiber $\pi^{-1}(x)$.
- (5) The map \mathbf{H} is linear on each fiber and satisfies the following axiom

$$(\forall v \in \pi^{-1}(x))(\mathbf{H}v = xv)$$

- (6) The maps \mathbf{a} and \mathbf{a}^\dagger are linear and move between fibers according to the following axiom:

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow (\exists v' \in \pi^{-1}(x+1))(\exists y \in k)(y^2 = x \wedge \mathbf{a}^\dagger v = yv' \wedge \mathbf{a}v' = yv))$$

Let $(L, k) \models T_{A_1}$. Then L is a ‘line bundle’ over the base k , though we do not claim local triviality. Each fiber $\pi^{-1}(x)$ is an x -eigenspace for \mathbf{H} . The elements $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$ are to be regarded as normal basis elements of the fiber $\pi^{-1}(x)$ which can be permuted by the group of l -th roots of unity $\Gamma[l]$. This setup serves as a discrete (and algebraic) model for a well-known phenomenon encountered when dealing with normed vector spaces. If V is a normed vector space over \mathbb{C} and $v \in V$ is an element of norm 1, then so is αv for any $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$. Of course, in this case there is an infinite group S^1 acting on the elements in V of norm 1. It will be seen, in the next chapter, that being able to permute the normal basis elements by a finite group is also essential for a decent structure theory for T_{A_1} .

Proposition 2.2. *Let $(L, k) \models T_{A_1}$. Then (L, k) is a representation of $A_1(k)$.*

Proof. Let $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e, x)$ holds. Then there is $e' \in \pi^{-1}(x+1)$ such that

$$\mathbf{H}\mathbf{a}^\dagger e = y\mathbf{H}e' = (x+1)ye' \quad y^2 = x$$

But

$$(x+1)ye' = (x+1)\mathbf{a}^\dagger e = \mathbf{a}^\dagger \mathbf{H}e + \mathbf{a}^\dagger e$$

Thus $[\mathbf{H}, \mathbf{a}^\dagger]e = \mathbf{a}^\dagger e$. Similarly, we obtain that $[\mathbf{H}, \mathbf{a}]e = -\mathbf{a}e$. Now

$$\mathbf{a}\mathbf{a}^\dagger e = y\mathbf{a}e' = xe$$

whereas

$$\mathbf{a}^\dagger \mathbf{a}e = z\mathbf{a}^\dagger e'' = (x-1)e$$

where $z^2 = x-1$ and $e'' \in \pi^{-1}(x-1)$. Thus $[\mathbf{a}, \mathbf{a}^\dagger]e = e$ as required. \square

2.2. $U_q(\mathfrak{sl}_2(k))$ for generic q . Let k be an algebraically closed field of characteristic 0, $q \in k$ with $q \neq 0, \pm 1$. The **quantized enveloping algebra** of $\mathfrak{sl}_2(k)$, denoted $U_q(\mathfrak{sl}_2(k))$, is defined to be the k -algebra with generators $E, F, K^{\pm 1}$ subject to the following relations

$$KEK^{-1} = q^2E \quad KFK^{-1} = q^{-2}F \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

along with $KK^{-1} = K^{-1}K = 1$. We associate a structure to this algebra when q is a generic parameter; namely when q is not a root of unity.

Definition 2.2. *Consider the two-sorted language $\mathcal{L}_q = (k, L, \pi, \mathbf{E}, E, F, K^{\pm 1}, q)$ where*

- (1) $\pi : L \rightarrow k$ and $E, F, K^{\pm 1} : L \rightarrow L$ are maps.
- (2) $\mathbf{E} \subseteq L \times k^*$ is a relation.
- (3) q is a constant from the sort k .

The sorts k, L are equipped with the same language as in condition 3 of Definition 2.1. The first-order \mathcal{L}_q -theory T_q states the following:

- (1) k is an algebraically closed field of characteristic 0.
- (2) $q^n \neq 1$ for every $n \in \mathbb{N}$.
- (3) The map $\pi : L \rightarrow k^*$ is surjective and each fiber $\pi^{-1}(x)$ for $x \in k^*$ is a one-dimensional k -vector space.
- (4) For each $x \in k^*$, the set $\mathbf{E}(L, x)$ is non-empty and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.
- (5) Fix an $l \in \mathbb{Z}$, $l > 0$, l even. If $\Gamma[l]$ denotes the group of l -th roots of unity of k , then there is a free and transitive actions of $\Gamma[l]$ on $\mathbf{E}(L, x)$ induced by the vector space action on the fiber $\pi^{-1}(x)$.

(6) The $K^{\pm 1}$ act on each fiber according to the following axiom:

$$(\forall v \in \pi^{-1}(\bar{x}))(Kv = xv \wedge K^{-1}v = x^{-1}v)$$

The maps $K^{\pm 1}$ are linear.

(7) The linear maps E and F move between the fibers according to the following axiom:

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow \begin{aligned} &(\exists v' \in \pi^{-1}(q^2x)(\exists y \in k) \\ &(y^2 = x \wedge Ev = \lambda(y)v' \wedge Fv' = -\lambda(qx)v)) \end{aligned})$$

where $\lambda : L \rightarrow k$ is defined by

$$\lambda(x) = \frac{y^{-1} + y}{q - q^{-1}}$$

If $(L, k) \models T_q$ then each fiber $\pi^{-1}(x)$ is an eigenspace for K with eigenvalue x . Each eigenspace contains a finite set of normal basis elements selected by \mathbf{E} , and permuted by $\Gamma[l]$.

Proposition 2.3. *Let $(L, k) \models T_q$. Then (L, k) is a representation of $U_q(\mathfrak{sl}_2(k))$.*

Proof. Consider $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e, x)$ holds. Then there are y such that $y^2 = x$ and $e' \in \pi^{-1}(q^2x)$ such that

$$KEe = \lambda(y)Ke' = \lambda(y)q^2xe'$$

But

$$EKe = xEe = x\lambda(y)e'$$

Thus $KEe = q^2EKe$. Similarly $KFe = q^{-2}FKe$. We shall now adopt the more intuitive notation $x^{1/2}$ for the element y such that $Ee = \lambda(y)e'$. Thus

$$Ee = \frac{x^{-1/2} + x^{1/2}}{q - q^{-1}}e'$$

and by the linearity of F ,

$$FEe = -\frac{(x^{-1/2} + x^{1/2})(q^{-1}x^{-1/2} + qx^{1/2})}{(q - q^{-1})^2}e$$

Whereas applying F first,

$$Fe = -\frac{x^{-1/2} + x^{1/2}}{q - q^{-1}}e'' \quad e'' \in \pi^{-1}(q^{-2}x)$$

hence

$$EFE = -\frac{(x^{-1/2} + x^{1/2})(qx^{-1/2} + q^{-1}x^{1/2})}{(q - q^{-1})^2}e$$

After some expansion and rearrangement,

$$(EF - FE)e = \frac{x - x^{-1}}{q - q^{-1}}e$$

as required. \square

2.3. Quantum torus. Our final example will be a certain multi-parameter quantum torus $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ where the parameters \mathbf{q} will depend on some generic q . Recall that this is the k -algebra with generators $\{\mathbf{U}_i^{\pm 1} : 1 \leq i \leq n\}$ subject to the relations

$$\mathbf{U}_i\mathbf{U}_i^{-1} = \mathbf{U}_i^{-1}\mathbf{U}_i = 1 \quad \mathbf{U}_i\mathbf{U}_j = q_{ij}\mathbf{U}_j\mathbf{U}_i \quad \text{for } i < j$$

We shall consider the following specific parameters

$$q_{ij} = q^{j-i}$$

which are best visualized as being upper triangular elements of a multiplicatively anti-symmetric $n \times n$ matrix:

$$\begin{pmatrix} * & q & q^2 & \dots & q^{n-1} \\ & * & q & \dots & q^{n-2} \\ \vdots & & \ddots & & q \\ & & & & * \end{pmatrix}$$

The base of our line bundle will parametrize eigenvalues of \mathbf{U}_1 and the remaining operators will move between fibers. We eliminate some linguistic preliminaries from the following definition, which should now be clear by referring to Definitions 2.1 and 2.2.

Definition 2.3. *We work with a two-sorted language $\mathcal{L}_q = (k, L, \pi, q, \mathbf{E}, \mathbf{U}_i^{\pm 1} : 1 \leq i \leq n)$. The \mathcal{L}_q -theory T_q says the following:*

- (1) k is an algebraically closed field of characteristic 0.
- (2) $q^n \neq 1$ for every $n \in \mathbb{N}$.
- (3) $\pi : L \rightarrow k^*$ is a surjective map and each fiber $\pi^{-1}(x)$ is a one-dimensional vector space for $x \in k$.
- (4) For each $x \in k$, $\mathbf{E}(L, x)$ is non-empty and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.
- (5) Let l be any positive integer (not necessarily odd). Then there is a free and transitive action of $\Gamma[l]$ on each $\mathbf{E}(L, x)$ induced by the vector space structure on $\pi^{-1}(x)$.
- (6) $\mathbf{U}_1^{\pm 1}$ are linear and we have

$$(\forall v \in \pi^{-1}(x))(\mathbf{U}_1 v = xv \wedge \mathbf{U}_1^{-1} v = x^{-1}v)$$

- (7) The linear maps $\mathbf{U}_2^{\pm 1}, \dots, \mathbf{U}_n^{\pm 1}$ move between fibers according to

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow \bigwedge_{i=2}^n (\exists v_i \in \pi^{-1}(q^{i-1}x))(\mathbf{E}(v, q^{i-1}x) \wedge \mathbf{U}_i v = xv_i \wedge \mathbf{U}_i^{-1} v_i = x^{-1}v))$$

- (8) For each $i < j$ with $i \neq 1$ we have the following axiom:

$$(\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow \mathbf{U}_i \mathbf{U}_j v = q^{j-i} \mathbf{U}_j \mathbf{U}_i v)$$

There are some points of difference with the previous examples worth noting. The first is that we have allowed our group $\Gamma[l]$ to be finite and cyclic of any order. The reasons for this are again model-theoretic. The second point is axiom 8 stipulating explicitly some good behaviour of basis elements with respect to the relations satisfied. This good behaviour was actually coded into the definitions of how the operators act between fibers in the previous two examples.

Proposition 2.4. *Let $(L, k) \models T_q$. Then (L, k) is a representation of $\mathcal{O}_q((k^\times)^n)$.*

Proof. Let $e \in \pi^{-1}(x)$ for some $x \in k^*$. By the axioms there is a $e_i \in \pi^{-1}(q^{i-1}x)$ such that $\mathbf{U}_i e = x e_i$. Thus

$$\mathbf{U}_1 \mathbf{U}_i e = x \mathbf{U}_1 e_i = x^2 q^{i-1} e_i$$

whereas

$$\mathbf{U}_i \mathbf{U}_1 e = x \mathbf{U}_i e = x^2 e_i$$

But $q^{i-1} = q_{1i}$, thus $\mathbf{U}_1 \mathbf{U}_i e = q_{1i} \mathbf{U}_i \mathbf{U}_1 e$. For $i < j$ and $i \neq 1$, we have $\mathbf{U}_i \mathbf{U}_j e = q^{j-i} \mathbf{U}_j \mathbf{U}_i e$ by definition. \square

Remark 2.1. *Note that for $i < j$ and $i \neq 1$, if $\mathbf{E}(e, x)$ holds then*

$$\mathbf{U}_i \mathbf{U}_j e = x \mathbf{U}_i e_j = q^{j-1} x^2 e_{ji}$$

for some $e_j \in \pi^{-1}(q^{j-1}x)$ and $e_{ji} \in \pi^{-1}(q^{i+j-2}x)$. On the other hand

$$\mathbf{U}_j \mathbf{U}_i e = x \mathbf{U}_j e_i = q^{i-1} x^2 e_{ij}$$

where $e_i \in \pi^{-1}(q^{i-1}x)$ and $e_{ij} \in \pi^{-1}(q^{i+j-2}x)$. Thus the stipulation that $\mathbf{U}_i \mathbf{U}_j e = q^{j-i} \mathbf{U}_j \mathbf{U}_i e$ implies that $e_{ij} = e_{ji}$.

3. EQUIVARIANT ALGEBRAS

We now define the class of equivariant algebras over an algebraically closed field k of characteristic 0. As indicated in the introduction, there are two parts to this definition. Firstly, we have to isolate a suitable class of algebras with an algebraic characterization, a class which we call ‘semi-equivariant’. An equivariant algebra is then defined to be a semi-equivariant algebra satisfying some additional (though rather cumbersome) assumptions. The reader is referred to Appendix B for basic definitions and notations concerning Hopf algebras.

Definition 3.1. *A prime k -algebra A is said to be **semi-equivariant** if*

- (1) *There is a maximal commutative affine k -subalgebra H of A that is a Hopf algebra.*
- (2) *A is generated as a k -algebra by the generators of H and finitely many eigenvectors $\mathbf{U}_1, \dots, \mathbf{U}_n$ of the left adjoint action of H on A ; namely the action defined by*

$$h \cdot a = \sum_{(h)} h' a S(h'') \quad \text{for all } a \in A \text{ and } h \in H$$

- (3) *There are generators h_1, \dots, h_m of H such that A has a presentation in terms of the h_i and \mathbf{U}_j and finitely many relations between them. All relations not expressing the adjoint action of h_i on \mathbf{U}_j have the form*

$$c \prod_{k=0}^{p-1} \mathbf{U}_{i_{p-k}} - d \prod_{k=0}^{q-1} \mathbf{U}_{i_{q-k}} = f(h_1, \dots, h_m)$$

where $c, d \in k$ and f is a polynomial over k .

If H and \mathbf{U}_i exist as above, then A is said to be semi-equivariant with respect to H and the elements $\mathbf{U}_1, \dots, \mathbf{U}_n$. It should be noted that although one typically defines the adjoint action of a Hopf algebra on itself, the definition makes sense in the current setting, thus turning A into a H -module.

Definition 3.1 takes its inspiration from the basic result that a finite dimensional complex semisimple Lie algebra \mathfrak{g} possesses a Cartan decomposition (see Appendix A). Recall that if \mathfrak{g} is such a Lie algebra then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where \mathfrak{h} is an abelian Lie subalgebra (the Cartan subalgebra) and the \mathfrak{g}_α are eigenspaces of the adjoint action of \mathfrak{g} on itself. Regarding $U(\mathfrak{g})$ (the universal enveloping algebra of \mathfrak{g}) as a Hopf algebra, the Lie algebra and Hopf algebra adjoint actions agree. Thus, heuristically, Definition 3.1 says that a semi-equivariant algebra A has a ‘generalized Cartan decomposition’ where the eigenvectors of the adjoint action satisfy some manageable relations amongst themselves.

3.1. Towards equivariant algebras. Let A be a semi-equivariant k -algebra. Because A is prime, H is a domain and by the assumption that H is an affine k -algebra, it is therefore the coordinate ring of an affine variety V . Suppose that $V \subseteq k^m$. Clearly there is a bijective correspondence between points of V and characters (k -algebra homomorphisms) on H , and we denote the character corresponding to $x \in V$ by χ_x . Let L_x be a one-dimensional k -vector space endowed with the structure of a H -module by the character χ_x , i.e.

$$hv = \chi_x(h)v \quad \text{for all } h \in H \text{ and } v \in L_x$$

The H -modules L_x will form the fibers of a line bundle over V ; thus we form the disjoint union

$$L = \coprod_{x \in V} L_x$$

and define the surjective map $\pi : L \rightarrow V$ by $\pi(v) = x$ if $v \in L_x$.

Lemma 3.1. *V is a group.*

Proof. This is a consequence of H being a Hopf algebra. First note that if $\chi_x, \chi_y : H \rightarrow k$ are the characters corresponding to the points $x, y \in V$ then $\chi_x \otimes \chi_y$ is a character on $H \otimes H$ by

$$(\chi_x \otimes \chi_y)(h_1 \otimes h_2) = \chi_x(h_1)\chi_y(h_2) \quad h_1, h_2 \in H$$

extended to an algebra homomorphism. Now $\chi = (\chi_x \otimes \chi_y) \circ \Delta$ is a character on H and the kernel of χ is a maximal ideal of H , thus corresponding to a point $z \in V$. This allows us to define a map $\cdot : (y, z) \mapsto z$. The coassociativity of Δ easily implies that \cdot is associative. Similarly, we define $\chi_{x^{-1}} = \chi_x \circ S$ and put x^{-1} as the point in V corresponding to the kernel of $\chi_{x^{-1}}$. \square

Because A is semi-equivariant, we have

$$h \cdot \mathbf{U}_i = \chi_i(h) \mathbf{U}_i \quad \text{for all } h \in H$$

for some characters $\chi_i : H \rightarrow k$. In particular, we obtain constants $\alpha_{ji} = \chi_i(h_j)$ such that $h_j \cdot \mathbf{U}_i = \alpha_{ji} \mathbf{U}_i$ for all $1 \leq j \leq m$.

Lemma 3.2. *Let $v \in L_x$. Then*

$$\sum_{(h)} \chi_{x^{-1}}(h_j'') h_j' \mathbf{U}_i v = \alpha_{ji} \mathbf{U}_i v$$

Proof. By $h_j \cdot \mathbf{U}_i = \alpha_{ji} \mathbf{U}_i$ we obtain that

$$\sum_{(h)} h_j' \mathbf{U}_i S(h_j'') v = \alpha_{ji} \mathbf{U}_i v$$

But $S(h_j'')$ acts on v by the scalar $\chi_{x^{-1}}(h_j'')$ and the result follows. \square

Corollary 3.1. *Suppose that for each $1 \leq j \leq m$, $\Delta(h_j) = \sum_{(h)} h_j' \otimes h_j''$ is such that*

- (1) $h_j' = h_j$ or 1 for every element h_j' in the sum.
- (2) There is at least one h_j' with $h_j' = h_j$.

Then there are regular maps $r_j, s_j : V \rightarrow k$ such that for each $v \in L_x$,

$$h_j \mathbf{U}_i v = \frac{\alpha_{ji} - r_j(x)}{s_j(x)} \mathbf{U}_i v$$

Proof. Immediate from Lemma 3.2. \square

Assuming that the coproduct satisfies the assumptions of Corollary 3.1, we have functions $\eta_{ji}(x) = (\alpha_{ji} - r_j(x))/s_j(x)$. Given $x \in V$, put $\eta_i(x) = (\eta_{1i}(x), \dots, \eta_{mi}(x))$ for each $1 \leq i \leq n$. Then for each i , $\eta_i : V \rightarrow V$ because $\eta_i(x)$ defines a character on H for each $x \in V$. We obtain that $\mathbf{U}_i : L_x \rightarrow L_{\eta_i(x)}$ for every $x \in V$ and $1 \leq i \leq n$. Let Π be the semigroup generated by the η_i (under composition). Thus Π acts on V in an obvious way. We shall, henceforth, treat the elements of Π as functions on V and adopt the usual convention for composition of functions; namely for $\eta_1, \eta_2 \in \Pi$, $\eta_1 \eta_2 = \eta_1 \circ \eta_2$ (apply η_2 first, then η_1).

We can now define the class of equivariant algebras. As stated previously, the definition has the sole purpose of narrowing down the class of semi-equivariant algebras to those which have a suitably geometric first-order definable space which is a representation of A .

Definition 3.2. *We define a semi-representable algebra A to be **equivariant** if*

- (1) *For each $1 \leq j \leq m$, $\Delta(h_j) = \sum_{(h)} h_j' \otimes h_j''$ is such that*
 - (a) $h_j' = h_j$ or 1 for each element in the sum.
 - (b) $h_j' = h_j$ for at least one h_j' .
- (2) Π *is a group such that for each $1 \leq i \leq n$, $\eta_i^{-1} = \eta_j$ for some $j \leq n$.*
- (3) V *is defined over \mathbb{Q} .*
- (4) *There exist regular functions $\lambda_i : V \rightarrow k$ and polynomials*

$$P_i(x, y) := y^{n_i} - \mu_i(x) \quad n_i \in \mathbb{N}, n_i > 0 \quad \mu_i \in \Pi$$

for each $1 \leq i \leq n$ such that for each relation of the form

$$(1) \quad c \prod_{k=0}^{p-1} \mathbf{U}_{i_{p-k}} - d \prod_{k=0}^{q-1} \mathbf{U}_{j_{q-k}} = f(h_1, \dots, h_m)$$

we have

- (a) $\eta_{i_p} \dots \eta_{i_1} = \eta_{j_q} \dots \eta_{j_1}$ and $\eta_{j_q} \dots \eta_{j_1} = 1$ if $f \neq 0$.

(b)

$$(2) \quad c \prod_{k=0}^{p-1} \lambda_{i_{p-k}}(y_{i_{p-k}}) - d \prod_{k=0}^{q-1} \lambda_{j_{q-k}}(y_{j_{q-k}}) = f(x)$$

holding for every $x \in V$, where

- (i) y_{i_1} (respectively y_{j_1}) is a solution to the polynomial equation $P_{i_1}(x, y_{i_1}) = 0$ (respectively $P_{j_1}(x, y_{j_1}) = 0$).
- (ii) $y_{i_{p-k}}$ (respectively $y_{j_{q-k}}$) for $k < p-1$ is a solution to the polynomial equation

$$P_{i_{p-k}}\left(\left(\prod_{r>k} \eta_{i_{p-r}}\right)(x), y_{i_{p-k}}\right) = 0 \quad \left(P_{j_{q-k}}\left(\left(\prod_{r>k} \eta_{j_{q-r}}\right)(x), y_{j_{q-k}}\right) = 0 \right)$$

Moreover, the roots y_{i_k} can be chosen compatibly for all conditions of the form (4.2) and for all $x \in V$.

- (5) The parameters appearing in all λ_i, η_i and f , along with the constants c, d , are solutions to types over \mathbb{Q} .
- (6) The maps λ_i are Γ -linear, where Γ is the group of roots of unity of order l for some $l > 0$ such that $n_i | l$ for all $1 \leq i \leq n$.

The significance of many of these conditions will only become apparent when we commence doing some model theory, although 4 can be presently motivated. If an algebra A is semi-equivariant then the relations of the form (4.1) may not have much to do with the adjoint action of H on A . Condition 4 effectively remedies this. We already know that $\mathbf{U}_i : L_x \rightarrow L_{\eta_i(x)}$ for every $x \in V$, $1 \leq i \leq n$. So 4 (combined with 5) states that we can define the action of the \mathbf{U}_i , respecting the way they move between fibers, in such a way that all of these relations are satisfied regardless of what fiber L_x we start at. The use of the polynomials P_i and the functions λ_i is to allow a certain amount of definitional flexibility. For two of the preliminary examples considered (the Weyl algebra and $U_q(\mathfrak{sl}_2(k))$), this extra flexibility is necessary. We now define the category $\mathbf{Equiv}(k)$.

Definition 3.3. Let A and B be two equivariant k -algebras. A k -algebra homomorphism $\varphi : A \rightarrow B$ is defined to be **equivariant** if for any Hopf algebra H such that A is equivariant with respect to H and elements $\mathbf{U}_{11}, \dots, \mathbf{U}_{1n_1}$ of A , there is a Hopf subalgebra H' of B such that

- (1) B is equivariant with respect to H' and $\mathbf{U}_{21}, \dots, \mathbf{U}_{2n_2} \in B$.
- (2) $\varphi(\mathbf{U}_{1i})$ is a monomial in $\mathbf{U}_{21}, \dots, \mathbf{U}_{2n_2}$ for each i .
- (3) $\varphi|_H : H \rightarrow H'$.

Thus the category $\mathbf{Equiv}(k)$ is defined to consist of equivariant k -algebras and equivariant morphisms. We conclude this subsection with the following observation.

Proposition 3.1. Let A be semi-equivariant. Then there is an equivariant A' such that A is an epimorphic image of A' .

Proof. Say A is equivariant with respect to H (generated by h_1, \dots, h_m) and $\mathbf{U}_1, \dots, \mathbf{U}_n$. Define A' to have the same generators, to satisfy the relations of A expressing the adjoint action of H on the \mathbf{U}_i , and only those additional relations that satisfy equivariance. The result is now immediate. \square

3.2. Application to initial examples. We now show that our initial examples are equivariant algebras.

3.2.1. First Weyl algebra. We consider $A = A_1(k)$. Put $H = k[\mathbf{H}]$ and endow H with the Hopf algebra structure associated to universal enveloping algebras, namely

$$\Delta(\mathbf{H}) = 1 \otimes \mathbf{H} + \mathbf{H} \otimes 1 \quad \epsilon(\mathbf{H}) = 0 \quad S(\mathbf{H}) = -\mathbf{H}$$

The variety corresponding to H is the affine line k . For each $x \in k$, we have the one-dimensional H -module L_x given by the character $\chi_x : H \rightarrow k$ with $\chi_x(\mathbf{H}) = x$. Of course, the module L_x is merely an x -eigenspace for \mathbf{H} . It is easy to see that V is the group $(k, +)$. Indeed

$$\chi_{xy}(\mathbf{H}) = [(\chi_x \otimes \chi_y) \circ \Delta](\mathbf{H}) = \chi_x(1)\chi_y(\mathbf{H}) + \chi_x(\mathbf{H})\chi_y(1) = x + y$$

and

$$\chi_{x^{-1}}(\mathbf{H}) = -\chi_x(\mathbf{H}) = -x$$

For any $a \in A$,

$$\mathbf{H} \cdot a = \mathbf{H}a - a\mathbf{H} = [\mathbf{H}, a]$$

But $[\mathbf{H}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger$ and $[\mathbf{H}, \mathbf{a}] = -\mathbf{a}$, thus \mathbf{a}^\dagger and \mathbf{a} are eigenvectors for the adjoint action of H on A . Moreover, we have the additional relation

$$\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1$$

which is of the required form. Hence A is semi-equivariant. By inspection, the coproduct has the required form. Now we determine Π . Suppose that $v \in L_x$. By Corollary 3.1 we obtain

$$\mathbf{H}\mathbf{a}^\dagger v = (x+1)\mathbf{a}^\dagger v \quad \mathbf{H}\mathbf{a}v = (x-1)\mathbf{a}v$$

Thus the semigroup Π is generated by two functions:

$$\eta_\dagger(x) = x+1 \quad \eta(x) = x-1$$

Hence $\Pi = \mathbb{Z}$ is a group such that the inverse of η_\dagger is η (and vice versa). Furthermore, the parameters appearing η_\dagger and η are integral. For the relation $[\mathbf{a}, \mathbf{a}^\dagger] = 1$, we note that for every $x \in k$,

$$(x^{1/2})^2 - ((x-1)^{1/2})^2 = 1$$

where $x^{1/2}$ denotes some y such that $y^2 = x$. So we put $\lambda(y) = \lambda_\dagger(y) = y$ and

$$P(x, y) = y^2 - \eta(x) \quad P_\dagger(x, y) = y^2 - x = P(\eta(x), y)$$

It is then clear that all of the roots y of P, P_\dagger can be chosen compatibly for all $x \in k$ (for any $x \in k$ we just pick, once and for all, any y such that $P(x, y) = 0$ and everything works). We can take Γ to be the group of roots of unity of order l for any even l . Trivially, λ_1 and λ_2 are Γ -linear.

3.2.2. $U_q(\mathfrak{sl}_2(k))$. Put $A = U_q(\mathfrak{sl}_2(k))$ and consider $H = k[K, K^{-1}]$. Then $V = k^*$ (which is definable over \mathbb{Q}) and we endow H with the group Hopf algebra structure, namely

$$\Delta(K) = K \otimes K \quad \epsilon(K) = 1 \quad S(K) = K^{-1}$$

with analogous relations for K^{-1} . For $x \in k^*$, L_x is therefore an x -eigenspace for K . Now V is the group (k^*, \cdot) . To see this, note that

$$\chi_{xy} = [(\chi_x \otimes \chi_y) \circ \Delta](K) = \chi_x(K)\chi_y(K) = xy$$

and

$$\chi_{x^{-1}} = \chi_x(K^{-1}) = x^{-1}$$

Now for any $a \in A$,

$$K \cdot a = KaK^{-1}$$

But $KEK^{-1} = q^2E$ and $KFK^{-1} = q^{-2}F$, thus E and F are eigenvectors of the adjoint action of H on A . We have the additional relation

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

which is of the required form, thus giving that A is semi-equivariant. Let $v \in L_x$. Then by Corollary 3.1,

$$KEv = q^2xEv \quad KFv = q^{-2}xFv$$

Thus Π is generated by the functions

$$\eta_E(x) = q^2x \quad \eta_F(x) = q^{-2}x$$

hence $\Pi = q^{2\mathbb{Z}} = \{q^l : l \in 2\mathbb{Z}\}$ (also a group) and η_E, η_F are mutually inverse. The parameter q appearing in the definition of η_E and η_F satisfies the type

$$\{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$$

By reference to Proposition 2.3, we take

$$\lambda_E(y) = -\lambda_F(y) = \frac{y^{-1} + y}{q - q^{-1}}$$

and

$$P_E(x, y) = P_F(x, y) = y^2 - x$$

By the calculation performed in Proposition 2.3, we obtain

$$\lambda_E(y_2)\lambda_F(y_1) - \lambda_F(z_2)\lambda_E(z_1) = \frac{x - x^{-1}}{q - q^{-1}}$$

for appropriate y_i, z_i . By contrast with the previous example, not any y_i, z_i will do and we have to be careful about picking them compatibly. For this purpose, we partition k^* into cosets of $q^{2\mathbb{Z}}$:

$$k^* = \bigcup_{x \in \Lambda} q^{2\mathbb{Z}}x$$

where Λ is a set of representatives. Given $x \in \Lambda$, choose any square root y of x . For any other $z \in q^{2\mathbb{Z}}x$, there is $l \in \mathbb{Z}$ such that $z = q^{2l}x$ and we choose the square root $q^l y$ of z . Now repeat this for every coset representative in Λ . The result is a compatible set of square roots. Clearly our polynomial P_E satisfies the required conditions involving parameters. Consider $\Gamma = \{\pm 1\}$. Then λ_E is Γ -linear, for

$$\lambda_E(-y) = \frac{-y^{-1} - y}{q - q^{-1}} = -\lambda_E(y)$$

hence so is λ_F .

3.2.3. The multiparameter quantum torus. Put $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$. We take the same Hopf algebra H as for $U_{\mathbf{q}}(\mathfrak{sl}_2(k))$, hence we obtain a line bundle L over the base (k^*, \cdot) . Again, for any $a \in A$ we have

$$\mathbf{U}_1 \cdot a = \mathbf{U}_1 a \mathbf{U}_1^{-1}$$

But $\mathbf{U}_1 \mathbf{U}_i \mathbf{U}_1^{-1} = q^{i-1} \mathbf{U}_i$ for $i > 1$. Moreover, the remaining relations are

$$\mathbf{U}_i \mathbf{U}_j - q^{j-i} \mathbf{U}_j \mathbf{U}_i = 0 \quad i < j$$

giving that A is semi-equivariant. By Corollary 3.1, for $v \in L_x$,

$$\mathbf{U}_1 \mathbf{U}_i v = x q^{i-1} \mathbf{U}_i v$$

Thus we have functions

$$\eta_i(x) = q^{i-1}x \quad i > 1$$

and Π is generated by these η_i and their inverses (hence Π is a group). Again, the single parameter q appearing in the definition of the η_i is generic, hence satisfies $\{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$. We take $\lambda_i(y) = y$ and $P_i(x, y) = y - x$ for all $i > 1$ and obtain what is required. On this occasion, the P_i are linear and we do not have to worry about roots. Because the λ_i are also linear, they are automatically Γ -linear for any group of roots of unity Γ .

3.3. Associating a theory to an equivariant algebra. Let A be equivariant. First, we establish some notation. Let $\mathbf{i} \in n^{<\omega}$ be a finite sequence of elements from $\{1, \dots, n\}$. Say $\mathbf{i} = (i_1, \dots, i_p)$. Put

$$\eta_{\mathbf{i}} = \eta_{i_p} \dots \eta_{i_1} \quad \mathbf{U}_{\mathbf{i}} = \mathbf{U}_{i_p} \dots \mathbf{U}_{i_1}$$

Definition 3.4. We consider the three-sorted language

$$\mathcal{L}_A = \{L, V, \Gamma, k, \pi, \mathbf{E}, \mathbf{U}_i, h_j, C : 1 \leq j \leq m, 1 \leq i \leq n\}$$

where

- (1) C is a finite set of constants.
- (2) $\pi : L \rightarrow V$ and $\mathbf{U}_i, h_j : L \rightarrow L$ are functions.
- (3) $\mathbf{E} \subseteq L \times V$ is a relation.
- (4) L, V, k are sorts, k has the language of rings and L comes equipped with
 - a map $+$: $L \times L \rightarrow k$ which is interpreted as addition of elements in each $\pi^{-1}(x)$ for $x \in V$.
 - a map \cdot : $k \times L \rightarrow L$ which is interpreted scalar multiplication in each fiber $\pi^{-1}(x)$.

The \mathcal{L}_A -theory T_A says the following:

- (1) k is an algebraically closed field of characteristic 0.

- (2) $\Sigma_c(c)$ holds for each $c \in C$, where Σ_c is the type that c satisfies.
- (3) $V = \phi(k)$ where ϕ is the formula over \mathbb{Q} defining V .
- (4) $\pi : L \rightarrow V$ is a surjective map and each fiber $\pi^{-1}(x)$ is a one-dimensional k -vector space for each $x \in V$.
- (5) $\mathbf{E}(L, x)$ is non-empty for each $x \in V$ and $\mathbf{E}(L, x) \subseteq \pi^{-1}(x)$.
- (6) Let Γ be the group of l -th roots of unity for some l satisfying condition 4 of the definition for representable algebras. Then Γ acts faithfully and transitively on $\mathbf{E}(L, x)$.
- (7) The operators h_j are linear and act on each fiber according to the following axiom:

$$(\forall v \in \pi^{-1}(x)) \left(\bigwedge_{j=1}^m h_j v = x_j v \right)$$

- (8) The linear operators \mathbf{U}_i act according to

$$(\forall v \in \pi^{-1}(x)) \exists v_i (\exists y_i \in k) (\mathbf{E}(v, x) \rightarrow \left(\bigwedge_{i=1}^n \mathbf{E}(v_i, \eta_i x) \wedge \mathbf{U}_i v = \lambda_i(y_i) v_i \wedge P_i(y_i, x) = 0 \right))$$

We denote the conjunct appearing in the big parentheses by $\varphi(v, v_i, y_i)$.

- (9) Enumerate the relations not expressing the adjoint action of H on the \mathbf{U}_i as

$$c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f_i(h_1, \dots, h_m) \quad 1 \leq i \leq r$$

where $\mathbf{j}_i, \mathbf{k}_i \in n^{<\omega}$ for each i . For any $\mathbf{i} = (i_1, \dots, i_p) \in n^{<\omega}$ we define the formula $\phi_{\mathbf{i}}(e_0, e, y)$ to be

$$\left(\bigwedge_{k=1}^p \mathbf{E}(e^k, \eta_{i_k} \pi(e^{k-1})) \wedge \mathbf{U}_{i_k} e^{k-1} = \lambda_{i_k}(y_k) e^k \wedge P_{i_k}(y_k, \pi(e^{k-1})) = 0 \right)$$

where $e^0 = e_0$. Then the following axiom holds:

$$\begin{aligned} (\forall v \in \pi^{-1}(x)) (\forall_{l=1}^r v^{(l)} \in \pi^{-1}(\eta_l x)) (\\ \mathbf{E}(v, x) \wedge \exists a^{(l)} \varphi(v, v^{(l)}, a^{(l)}) \rightarrow \quad & \forall_{i=1}^r v_i \forall_{i=1}^r w_i \forall_{i=1}^r y_i z_i (\bigwedge_{i=1}^r \phi_{\mathbf{j}_i}(v, v_i, y_i) \\ & \wedge \phi_{\mathbf{k}_i}(v, w_i, z_i) \wedge \theta_1 \wedge \psi) \wedge \\ & \forall_s v_s^{(l)} \forall_s w_s^{(l)} \forall_s y_s^{(l)} z_s^{(l)} (\bigwedge_{l=1}^r \bigwedge_{s=1}^r \phi_{\mathbf{j}_s}(v^{(l)}, v_s^{(l)}, y_s^{(l)}) \\ & \wedge \phi_{\mathbf{k}_s}(v^{(l)}, w_s^{(l)}, z_s^{(l)}) \wedge \theta_2)) \end{aligned}$$

where

- $\psi(y_i, z_i)$ is a conjunction of linear conditions which isolates the type $\text{tp}^k(y_i, z_i/\mathbb{Q} \cup C)$ formulated in the language of the sort k , for any instantiation of such y_i, z_i .
- θ_1 is the conjunction

$$\bigwedge_{\pi(v)=\pi(v_{ik})} v = v_{ik} \wedge \bigwedge_{\pi(v)=\pi(w_{ik})} v = w_{ik} \wedge \bigwedge_{\pi(v_{ik})=\pi(w_{jl})} v_{ik} = w_{jl}$$

and θ_2 is

$$\begin{aligned} \bigwedge_{\pi(v^{(l)})=\pi(v)} v^{(l)} = v \wedge \bigwedge_{\pi(v^{(l)})=\pi(v_{ik})} v^{(l)} = v_{ik} \wedge \bigwedge_{\pi(v^{(l)})=\pi(w_{ik})} v^{(l)} = w_{ik} \wedge \\ \bigwedge_{\pi(v_{st}^{(l)})=\pi(v_{ik})} v_{st}^{(l)} = v_{ik} \wedge \bigwedge_{\pi(w_{st}^{(l)})=\pi(w_{ik})} w_{st}^{(l)} = w_{ik} \end{aligned}$$

Axioms 8 and 9 of Definition 3.4 together express a significant amount of information. Whereas axiom 8 defines the action of each \mathbf{U}_i at a given fiber, axiom 9 is designed to make sure that all of the basis elements in different fibers obtained on repeated application of axiom 8, if they should agree, do agree. For example, given a defining relation of the form $c\mathbf{U}_{\mathbf{j}} - d\mathbf{U}_{\mathbf{k}} = f(h_1, \dots, h_m)$, one expects the basis elements involved in defining the action of $\mathbf{U}_{\mathbf{j}}$ and $\mathbf{U}_{\mathbf{k}}$ to eventually coincide in their respective terminal fibers. And if $f \neq 0$ then there should be more; namely that these terminal basis elements coincide with the basis element we start with. Only with such a stipulation is it possible to make sense of expressions like

$$c\mathbf{U}_{\mathbf{i}}e - d\mathbf{U}_{\mathbf{j}}e = f(x)e$$

where $e \in \pi^{-1}(x)$ is a basis element. In this manner, every model of T_A is indeed a representation of A . The formula ψ is a rigidity condition, ensuring that those roots of polynomials P_i which can be related to each other are indeed related to each other. In the initial examples discussed, ψ was implicitly incorporated into the axioms.

Example 3.1. Recall Definition 2.2 of the theory T_q associated to $U_q(\mathfrak{sl}_2(k))$. The actions of E and F were specified by

$$\begin{aligned} (\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow & (\exists v' \in \pi^{-1}(q^2x))(\exists y \in k) \\ & (y^2 = x \wedge Ev = \lambda(y)v' \wedge Fv' = -\lambda(qx)v)) \end{aligned}$$

where $\lambda : L \rightarrow k$ is defined by

$$\lambda(x) = \frac{y^{-1} + y}{q - q^{-1}}$$

Axiom 9 of Definition 3.4 would give

$$\begin{aligned} (\forall v \in \pi^{-1}(x))(\mathbf{E}(v, x) \rightarrow & (\forall v_1 \in \pi^{-1}(q^2x))(\forall v_2 \in \pi^{-1}(x))(\forall w_1 \in \pi^{-1}(q^{-2}x))(\forall w_2 \in \pi^{-1}(x)) \\ & (\forall_{i,j \leq 2} y_{ij} \in k)(y_{11}^2 = y_{21}^2 = x \wedge y_{12}^2 = q^2x \wedge y_{22}^2 = q^{-2}x \\ & \wedge Ev = \lambda(y_{11})v_1 \wedge Fv_1 = -\lambda(y_{12})v_2 \wedge Fv = -\lambda(y_{21})w_1 \\ & \wedge Ew_1 = \lambda(y_{22})w_2 \wedge v_2 = w_2 = v \wedge \psi) \end{aligned}$$

where ψ is chosen to be the formula $y_{12} = qy_{11} \wedge y_{11} = y_{21} \wedge y_{21} = qy_{22}$. With some simplification, this is indeed equivalent to the shorter axiom of Definition 2.2 when combined with axiom 8.

Remark 3.1. If A is equivariant, we may have some choice of possible λ_i, f and P_i . Nevertheless, in defining T_A a particular choice of these functions and polynomials is fixed once and for all. If one was being pedantic, the dependence of T_A on these functions and polynomials could have been indicated.

A model of T_A is therefore a three-sorted structure, which shall be denoted by a tuple (L, k) . We have suppressed V from the notation because it is evident that V is in fact definable in k . As stated in the introduction, such structures bear a striking resemblance to the G -equivariant line bundles (for G a connected algebraic group over \mathbb{C}) found in geometric representation theory (see [RTT07]). The proof of the following result takes its inspiration from the construction of the line bundle $L_\lambda = G \times_B \mathbb{C}_\lambda$, where B is a Borel subgroup of G and \mathbb{C}_λ is a one-dimensional representation of B corresponding to the weight λ of the Cartan subgroup H of G . The difference below is that we have $(\Gamma \times_\Gamma k) \times V$ instead. The resulting structure is then equipped with linear operators \mathbf{U}_i between fibers that give us some kind of equivariance, in the sense that $\mathbf{U}_i(L_x) = L_{\eta_i(x)}$ and $\mathbf{U}_i : L_x \rightarrow L_{\eta_i(x)}$ is a linear isomorphism, for each $x \in V$.

Proposition 3.2. T_A is consistent.

Proof. We construct a model of T_A . Let k be an algebraically closed field of characteristic 0. Introduce the equivalence relation

$$(\delta_1, a_1, x_1) \sim (\delta_2, a_2, x_2) \Leftrightarrow (\exists \gamma \in \Gamma)(a_2 = \gamma \cdot a_1 \wedge \delta_2 = \gamma^{-1}\delta_1)$$

on $\Gamma \times k \times V$ and consider the quotient $\Gamma \times k \times V / \sim$. We shall denote the equivalence class of (γ, a, x) by $\overline{(\gamma, a, x)}$. Put

$$L_x = \{\overline{(\gamma, a, x)} : \gamma \in \Gamma, a \in k\} \quad L = \prod_{x \in V} L_x$$

Then there is a projection map $\pi : L \rightarrow V$ given by $\pi(\overline{(\gamma, a, x)}) = x$. When L_x is understood, we suppress x from the notation and write $\overline{(\gamma, a)}$ for $\overline{(\gamma, a, x)}$.

Claim: Each L_x has the structure of a one-dimensional k -vector space by

$$\begin{aligned} \overline{(\delta_1, a_1)} + \overline{(\delta_2, a_2)} &:= \overline{(\delta_2, \gamma^{-1}a_1 + a_2)} \text{ where } \delta_2 = \gamma\delta_1 \\ \lambda(\overline{(\delta, a)}) &:= \overline{(\delta, \lambda a)} \end{aligned}$$

Proof. Suppose that $(\delta_1, a_1) \sim (\delta'_1, a'_1)$ and $(\delta_2, a_2) \sim (\delta'_2, a'_2)$ and that $\delta_2 = \gamma\delta_1$. There are γ_1, γ_2 such that $\delta'_1 = \gamma_1\delta_1$ and $\delta'_2 = \gamma_2\delta_2$. Thus

$$\delta'_2 = \gamma_2\gamma\gamma_1^{-1}\delta'_1$$

So it remains to prove that

$$(\delta_2, \gamma^{-1}a_1 + a_2) \sim (\delta'_2, \gamma_1\gamma^{-1}\gamma_2^{-1}a'_1 + a'_2)$$

But $\gamma_1^{-1}a_1 = a'_1$ and $\gamma_2^{-1}a_2 = a'_2$. So

$$\gamma_1\gamma\gamma_2^{-1}a'_1 + a'_2 = \gamma_2^{-1}(\gamma^{-1}a_1 + a_2)$$

as required. Scalar multiplication is trivially well-defined and $\overline{(1, 1)}$ is a basis element for L_x . \square

Normal basis elements are designated as those of the form $\overline{(\gamma, 1)}$ for $\gamma \in \Gamma$ and it is clear that Γ acts faithfully and transitively on the set of normal basis elements of L_x . We now define maps by

$$\mathbf{U}_i(\overline{(1, 1, x)}) = \lambda_i(y_i)\overline{(1, 1, \eta_i(x))} \quad 1 \leq i \leq n$$

and extend linearly. By condition 4 of Definition 3.2, the roots y_i can be chosen compatibly so that all listed relations of the form

$$c \prod_{k=0}^{p-1} \lambda_{i_p-k}(y_{i_p-k}) - d \prod_{k=0}^{q-1} \lambda_{j_q-k}(y_{j_q-k}) = f(x)$$

are satisfied for every $x \in V$. So we use these y_i when defining the actions of the \mathbf{U}_i . It is now clear that our resulting structure satisfies T_A . \square

We conclude this subsection with a remark about the types Σ_c . The theory T_A will only be adequate insofar as each Σ_c contains all of the information that is required of c . The examples considered above contained at most one constant q , and all that was required of q in these cases was that q was generic; namely that it satisfied the type $\Sigma_q = \{x^n \neq 1 : n \in \mathbb{N}, n > 0\}$.

4. MODEL THEORY OF EQUIVARIANT STRUCTURES: I

In this section and the next, we build on many of the results of [ZS09]. There it was proved that an uncountably categorical first-order theory can be associated to the Heisenberg algebra and a quantifier elimination result (down to the level of existential formulas) was established. Analogous results are proved here for T_A where A is an equivariant algebra with the property that models of T_A are, roughly speaking, rather rigid.

4.1. Categoricity. We shall fix an equivariant algebra A . As part of the definition, we have a certain amount of data (the regular functions λ_i , polynomials P_i and f , functions η_i). In the language \mathcal{L}_A , all these entities become definable over \mathbb{Q} and the constants C . We recall that each of these constants has its properties fixed by a type Σ_c over the prime subfield. For ease of reference we recall axiom 9 of Definition 3.4:

$$\begin{aligned} (\forall v \in \pi^{-1}(x))(\forall_{l=1}^r v^{(l)} \in \pi^{-1}(\eta_l x)) & \\ \mathbf{E}(v, x) \wedge \exists a^{(l)} \varphi(v, v^{(l)}, a^{(l)}) \rightarrow & \\ & \forall_{i=1}^r v_i \forall_{i=1}^r w_i \forall_{i=1}^r y_i z_i (\bigwedge_{i=1}^r \phi_{\mathbf{j}_i}(v, v_i, y_i) \\ & \wedge \phi_{\mathbf{k}_i}(v, w_i, z_i) \wedge \theta_1 \wedge \psi) \wedge \\ & \forall_s v_s^{(l)} \forall_s w_s^{(l)} \forall_s y_s^{(l)} z_s^{(l)} (\bigwedge_{l=1}^r \bigwedge_{s=1}^r \phi_{\mathbf{j}_s}(v^{(l)}, v_s^{(l)}, y_s^{(l)}) \\ & \wedge \phi_{\mathbf{k}_s}(v^{(l)}, w_s^{(l)}, z_s^{(l)}) \wedge \theta_2)) \end{aligned}$$

where

- The relations not expressing the adjoint action of H on the \mathbf{U}_i are enumerated as

$$c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f_i(h_1, \dots, h_m) \quad 1 \leq i \leq r$$

where $\mathbf{j}_i, \mathbf{k}_i \in n^{<\omega}$ for each i .

- For any $\mathbf{i} = (i_1, \dots, i_p) \in n^{<\omega}$ we define the formula $\phi_{\mathbf{i}}(e_0, e, y)$ to be

$$\left(\bigwedge_{k=1}^p \mathbf{E}(e^k, \eta_{i_k} \pi(e^{k-1})) \wedge \mathbf{U}_{i_k} e^{k-1} = \lambda_{i_k}(y_k) e^k \wedge P_{i_k}(y_k, \pi(e^{k-1})) = 0 \right)$$

where $e^0 = e_0$.

- $\psi(y_i, z_i)$ is a conjunction of linear conditions which isolates the type $\text{tp}^k(y_i, z_i/\mathbb{Q} \cup C)$ formulated in the language of the sort k , for any instantiation of such y_i, z_i .
- θ_1 and θ_2 combined express that those basis elements which should agree (i.e. those which lie in the same fibers), do agree.

In order for T_A to be categorical in uncountable cardinals, given any model $(L, k) \models T_A$ where k is an uncountable field, one requires the basis elements in the fibers above the orbit of any $x \in V$ under Π to remain rigid under scaling by certain elements of Γ . Specifically, axioms 8 and 9 provide us with a set of basis elements (one for each fiber) over the orbit Πx . The requirement is that the truth of axiom 9 should not be affected by shifting these basis elements by certain $\gamma \in \Gamma$. The following technical restriction is designed to achieve this. First some notation. Let Ξ consist of those pairs $(\mathbf{i}, \mathbf{j}) \in (n^{<\omega})^2$ selected by $\theta_1 \wedge \theta_2$ in axiom 9 with $\eta_{\mathbf{i}} = \eta_{\mathbf{j}}$, i.e. for any $x \in V$ and basis element $e \in \pi^{-1}(x)$, $\theta_1 \wedge \theta_2$ says that the basis elements used to define $\mathbf{U}_{\mathbf{i}}e$ and $\mathbf{U}_{\mathbf{j}}e$ lying in $\pi^{-1}(\eta_{\mathbf{i}}x)$ and $\pi^{-1}(\eta_{\mathbf{j}}x)$ respectively, agree.

Definition 4.1. *The theory T_A is Γ -rigid if for every model $(L, k) \models T_A$ and $(\mathbf{i}, \mathbf{j}) \in \Xi$ with $\mathbf{i} = (i_1, \dots, i_p)$, $\mathbf{j} = (j_1, \dots, j_q)$ we have that*

$$\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1 \Rightarrow \gamma^p = \delta^q$$

for every $\gamma, \delta \in \Gamma$.

Proposition 4.1. *Let T_A be Γ -rigid.*

- (1) *For every polynomial $P_i(x, y) = y^{n_i} - \mu_i(x)$, we have $n_i \leq 2$.*
- (2) *For every relation of the form $c\mathbf{U}_{\mathbf{i}} - d\mathbf{U}_{\mathbf{j}} = f(h_1, \dots, h_m)$, if $\mathbf{i} = (i_1, \dots, i_p)$ and $\mathbf{j} = (j_1, \dots, j_q)$ then one of the following holds:*
 - (a) $p = q$.
 - (b) $p = 2q$.
 - (c) $q = 2p$.

Proof. (1) A given $\eta_i \in \Pi$ has an inverse η_j and $(i, j) \in \Xi$. Thus for any $\gamma, \delta \in \Gamma$ such that $\gamma^{n_i} = \delta^{n_j} = 1$, it follows by Γ -rigidity (applied with $\eta_i \eta_j = \eta_j \eta_i$ and $\eta_i \eta_j = 1$) that $\gamma^2 = \delta^2 = 1$. In particular, this holds if γ and δ are primitive roots of unity. Thus $n_i, n_j \leq 2$.

(2) Suppose that $\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1$ where γ and δ are primitive. Then $\gamma^{qn_{i_1}} = \delta^{qn_{j_1}} = \gamma^{pn_{j_1}} = 1$ (by $\gamma^p = \delta^q$). Because γ is primitive, $qn_{i_1} \leq pn_{j_1}$. The reverse inequality follows by symmetry, hence $qn_{i_1} = pn_{j_1}$. The result is now immediate by 1. \square

Proposition 4.1 gives some indication of the strength of the assumption of Γ -rigidity. The following results show that Γ -rigidity is equivalent to uncountable categoricity.

Proposition 4.2. *Suppose that $(L, k) \models T_A$ witnesses the failure of Γ -rigidity, where k is uncountable. Then there is an automorphism σ of k which does not extend to an automorphism of (L, k) .*

Proof. There is $(\mathbf{i}, \mathbf{j}) \in \Xi$ such that for some basis element $e \in \pi^{-1}(x)$, there are e_1, e_2, y_1, y_2 with

$$(L, k) \models \phi_{\mathbf{i}}(e, e_1, y_1) \wedge \phi_{\mathbf{j}}(e, e_2, y_2)$$

but there are $\gamma, \delta \in \Gamma$ such that $\gamma^{n_{i_1}} = \delta^{n_{j_1}} = 1$, and $\gamma^p \neq \delta^q$ (where $\mathbf{i} = (i_1, \dots, i_p)$, $\mathbf{j} = (j_1, \dots, j_q)$). Let y'_1, y'_2 be tuples obtained by transforming $y_{1i} \mapsto \gamma y_{1i}$ and $y_{2i} \mapsto \delta y_{2i}$. Now ψ implies a formula $\psi_{\mathbf{i}, \mathbf{j}}$ which isolates the type $\text{tp}^k(y_1, y_2/\mathbb{Q} \cup C)$; indeed $\psi_{\mathbf{i}, \mathbf{j}}$ is just a suitable subformula of ψ . In particular it is a conjunction of linear conditions. Because the y_{1i} (respectively y_{2i}) are all related to each other via $\psi_{\mathbf{i}, \mathbf{j}}$, $\psi_{\mathbf{i}, \mathbf{j}}$ also holds of y'_1, y'_2 . Thus $\text{tp}_k(y_1, y_2/\mathbb{Q} \cup C) = \text{tp}_k(y'_1, y'_2/\mathbb{Q} \cup C)$. By saturation

of k , there is an automorphism σ of k such that $\sigma(y_1, y_2) = (y'_1, y'_2)$. Suppose for contradiction that σ does extend to an automorphism $\tilde{\sigma}$ of (L, k) . Decomposing $\phi_i(e, e_1, y_1)$ we obtain

$$(L, k) \models \mathbf{U}_i e = \prod_{k=1}^p \lambda_{i_k}(y_{1k}) e_1^p \Rightarrow (L, k) \models \mathbf{U}_i \tilde{\sigma}(e) = \prod_{k=1}^p \lambda_{i_k}(\sigma(y_{1k})) \tilde{\sigma}(e_1^p)$$

Now $\prod_{k=1}^p \lambda_{i_k}(\sigma(y_{1k})) = \gamma^p \prod_{k=1}^p \lambda_{i_k}(y_{1k})$. But $y_{11}^{n_{i_1}} = \mu_{i_1} x$, hence

$$\mu_{i_1} x = y_{11}^{n_{i_1}} = \gamma^{n_{i_1}} y_{11}^{n_{i_1}} = \mu_{i_1} \sigma(x) \Rightarrow x = \sigma(x)$$

the implication holding because Π is a group, hence $\tilde{\sigma}(e)$ and e lie in the same fiber and $\tilde{\sigma}(e) = \mu e$ for some $\mu \in k$. Now we apply the same argument to $\phi_j(e, e_2, y_2)$. By axiom 9, $e_1^p = e_2^p$. But then

$$\tilde{\sigma}(e_1^p) = \gamma^{-p} \mu e_1^p = \delta^{-q} \mu e_2^p = \tilde{\sigma}(e_2^p)$$

hence $\gamma^p = \delta^q$, resulting in contradiction. \square

Theorem 4.1. T_A is Γ -rigid if and only if for any models $(L, k), (L', k') \models T_A$ with k, k' uncountable, if $\sigma : k \rightarrow k'$ is an isomorphism then σ extends to an isomorphism $\tilde{\sigma} : L \rightarrow L'$.

Proof. Suppose that T_A is Γ -rigid. Because V is defined over the prime subfield, $\sigma : V \rightarrow V' = \phi(k')$. Similarly $\sigma : \Gamma \rightarrow \Gamma'$ where Γ' is the group of l -th roots of unity in k' . We can assume that σ maps constants to constants (Σ_c are over \mathbb{Q} , hence are preserved by σ). Because all of the information we require of a constant is contained in Σ_c , we may as well reinterpret the constants of (L', k') so that they lie in the image of $C(k)$ under σ). Partition V up into orbits of the group Π , thus obtaining

$$V = \bigcup_{x \in \Lambda} \Pi x$$

for some set of representatives Λ of each orbit. It is then clear that we have a corresponding partition for V'

$$V' = \bigcup_{\sigma(x) \in \sigma(\Lambda)} \Pi' \sigma(x)$$

where Π' is the group generated by $\eta'_i = \sigma(\eta_i)$. Define the **length** of y (with respect to the representative x), $l(y)$, to be the length of the smallest sequence \mathbf{i} such that $y = \eta_{\mathbf{i}}(x)$. We extend $\tilde{\sigma}$ to the rest of the orbit Πx by induction on length.

- (1) $l(y) = 0$, i.e. $y = x$. By axiom 5 of T_A there is $e \in \pi^{-1}(x)$ such that $\mathbf{E}(e, x)$ holds in (L, k) . Likewise, there is $e' \in \pi^{-1}(\sigma(x))$ such that $(L', k') \models \mathbf{E}(e', \sigma(x))$ and we define $\tilde{\sigma} : L_x \rightarrow L'_{\sigma(x)}$ by mapping $e \mapsto e'$ and extending linearly. By repeated application of axiom 8 we obtain basis elements e_{1i}, e_{2i} and elements y_{1i}, y_{2i} of k such that

$$(L, k) \models \bigwedge_{i=1}^r \phi_{\mathbf{j}_i}(e, e_{1i}, y_{1i}) \wedge \phi_{\mathbf{k}_i}(e, e_{2i}, y_{2i})$$

where $\mathbf{j}_i, \mathbf{k}_i$ are such that $c_i \mathbf{U}_{\mathbf{j}_i} - d_i \mathbf{U}_{\mathbf{k}_i} = f(h_1, \dots, h_m)$. Similarly we obtain basis elements e'_{1i}, e'_{2i} and $y'_{1i}, y'_{2i} \in k'$ such that

$$(L', k') \models \bigwedge_{i=1}^r \phi_{\mathbf{j}_i}(e', e'_{1i}, y'_{1i}) \wedge \phi_{\mathbf{k}_i}(e', e'_{2i}, y'_{2i})$$

where ψ' is ψ with all parameters from k transformed to their images under σ . Fix some $1 \leq i \leq r$. Suppose that on decomposing $\phi_{\mathbf{j}_i}(e, e_{1i}, y_{1i})$ we obtain

$$(L, k) \models \bigwedge_{k=1}^{p_i} \mathbf{E}(e_{1i}^k, \eta_{j_k} \pi(e_{1i}^{k-1})) \wedge \mathbf{U}_{j_k} e_{1i}^{k-1} = \lambda_{j_k}(y_{1ik}) e_{1i}^k \wedge P_{j_k}(y_{1ik}, \pi(e_{1i}^{k-1})) = 0$$

Put $P'_l = \sigma(P_l)$ for each $1 \leq l \leq n$. Note that $\eta'_l(\sigma(x)) = \sigma(\eta_l(x))$ for every such l and $x \in V$. We also have

$$(L', k') \models \bigwedge_{k=1}^{p_i} \mathbf{E}(e'_{1i}^k, \eta'_{j_k} \pi(e'_{1i}^{k-1})) \wedge \mathbf{U}_{j_k} e'_{1i}^{k-1} = \lambda'_{j_k}(y'_{1ik}) e'_{1i}^k \wedge P'_{j_k}(y'_{1ik}, \pi(e'_{1i}^{k-1})) = 0$$

Thus $P'_{j_k}(y'_{1ik}, \pi(e'^{k-1}_{1i})) = P'_{j_k}(\sigma(y_{1ik}), \pi(e'^{k-1}_{1i})) = 0$ for every k . Recalling that $P_{j_k}(y, x) = y^{n_{j_k}} - \mu_{j_k}(x)$ and that n_{j_k} divides the order of Γ , it follows that there are $\gamma_{1ik} \in \Gamma$ such that $y'_{1ik} = \gamma_{1ik}\sigma(y_{1ik})$ for every k . But then by Γ -linearity of the λ_{j_k} , it follows that

$$(L', k') \models \mathbf{U}_{j_k} e'^{k-1}_{1i} = \gamma_{1ik} \lambda'_{j_k}(\sigma(y_{1ik})) e'^k_{1i}$$

Thus we define $\tilde{\sigma}$ on the fibers containing the e^k_{1i} by mapping $e^k_{1i} \mapsto \gamma_{1ik} e'^k_{1i}$ and extending linearly. Repeat the above for $\phi_{\mathbf{k}_i}(e, e_{2i}, y_{2i})$ and every $1 \leq i \leq r$. By axiom 9, $\psi \in \text{tp}^k(y_{1i}, y_{2i}/\mathbb{Q} \cup C) \Rightarrow \psi' \in \text{tp}^k(\sigma(y_{1i}), \sigma(y_{2i})/\mathbb{Q} \cup C')$. Thus any roots of the various polynomials P_l related by ψ transform by the same element of Γ under σ . Now we invoke Γ -rigidity in (L', k') . Let $(\mathbf{i}, \mathbf{j}) \in \Xi$. Recall from the proof of Proposition 4.2 that there is a conjunction of linear conditions $\psi'_{\mathbf{i}, \mathbf{j}}$ implied by ψ' that relates all those roots used to define the action of $\mathbf{U}_{\mathbf{i}} e'$ (respectively $\mathbf{U}_{\mathbf{j}} e'$). Concentrating on $\mathbf{U}_{\mathbf{i}} e'$, for the sake of notational clarity, we assume that these roots are the first p elements of the tuple y'_{1i} . Thus there is γ_{1i} such that $\gamma_{1i} = \gamma_{1ik}$ for $1 \leq k \leq p$ and

$$(L', k') \models \mathbf{U}_{\mathbf{j}} e' = \prod_{k=1}^p \lambda'_{j_k}(y'_{1ik}) e'^p_{1i} = \gamma_{1i}^p \prod_{i=1}^p \lambda'_{j_k}(\sigma(y_{1ik})) e'^p_{1i}$$

By Γ -rigidity, $\gamma_{1i}^p = 1$. Because the γ_{1i} get successively absorbed under $\tilde{\sigma}$ we also have that $\mathbf{U}_{\mathbf{j}} e' = \prod_{k=1}^p \lambda'_{j_k}(\sigma(y_{1ik})) \tilde{\sigma}(e^p_{1i})$, hence $\tilde{\sigma}(e^p_{1i}) = e'^p_{1i}$. Exactly the same argument applies to $\mathbf{U}_{\mathbf{j}} e'$, hence applying $\tilde{\sigma}$ as defined still preserves the truth of axiom 9 in (L', k') , as required.

- (2) $l(y) > 0$; thus $\tilde{\sigma}$ has been extended to all those L_z for which $l(z) \leq l(y) - 1$. Let $\mathbf{l} = (l_1, \dots, l_s) \in n^{<\omega}$ witness the length of y . Put $z = \eta_{l_{s-1}} \dots \eta_{l_1} x$. Then $l(z) \leq l(y) - 1$ and by induction $\tilde{\sigma}$ has already been extended to $\pi^{-1}(z)$. Let $e^{(1)} \in \pi^{-1}(z)$ be the basis element used to define $\mathbf{U}_{l_{s-1}} \dots \mathbf{U}_{l_1} e$. Now apply axiom 8 to $\pi^{-1}(z)$ with $e^{(1)}$ to obtain $e^{(2)} \in \pi^{-1}(y)$ in terms of which $\mathbf{U}_{l_s} e^{(1)}$ is defined and extend as in the base case. Now repeat the whole argument for the base case with $e^{(2)}$ and the induction step follows.

Repeating the above construction for each orbit completes the construction of $\tilde{\sigma}$. The converse is immediate by Proposition 4.2. \square

Proposition 4.3. *The theories associated to $A_1(k)$, $U_q(\mathfrak{sl}_2(k))$ and $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ are Γ -rigid.*

Proof. We start with $A_1(k)$. Recall that for this algebra Π is generated by two functions

$$\eta_{\dagger}(x) = x + 1 \quad \eta(x) = x - 1$$

The only relation not expressing the adjoint action of \mathbf{H} is $\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1$, hence $\eta_{\dagger}\eta = \eta\eta_{\dagger}$. Now the corresponding polynomials are

$$P(x, y) = y^2 - \eta(x) \quad P_{\dagger}(x, y) = y^2 - x$$

so we must demonstrate that for any $\gamma, \delta \in \Gamma$, $\gamma^2 = \delta^2 = 1 \Rightarrow \gamma^2 = \delta^2$ which is trivially true! For $U_q(\mathfrak{sl}_2(k))$, Π is generated by

$$\eta_E(x) = q^2 x \quad \eta_F(x) = q^{-2} x$$

and similarly $\eta_E \eta_F = \eta_F \eta_E$. The corresponding polynomials are

$$P_E(x, y) = P_F(x, y) = y^2 - x$$

so again, Γ -rigidity trivially holds. Γ -rigidity for $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ is immediate because all of the polynomials involved are $P_i(x, y) = y - x$, which are linear. \square

4.2. Quantifier elimination. From now on, we assume that T_A is Γ -rigid. Firstly, we provide some motivation for the definable sets we wish to consider. Fix a model $(L, k) \models T_A$ and suppose that $v = (v_1, \dots, v_s)$ is a tuple from the sort L . We can re-index the v_i according to the fibers of π in which they appear. Namely, we fix an enumeration $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ so that given v_{ij}, v_{kl} , we have $i = k$ if and only if $\pi(v_{ij}) = \pi(v_{kl})$. By the axioms, we can find

m -tuples $a_i \in V \subseteq k^m$ such that $\pi(v_{ij}) = a_i$ for every $1 \leq i \leq t$. Moreover, because each $\pi^{-1}(a_i)$ is one-dimensional and we have basis elements $e_i \in \mathbf{E}(L, a_i)$, we can find scalars $\lambda_{ij} \in k$ such that

$$\bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \lambda_{ij} e_i = v_{ij}$$

holds in (L, k) . Thus one expects all the sentences satisfied by v to be determined by all the inter-relationships between the e_i . But the relationships between the e_i depend on the orbits of Π on V . We set up some notation to describe these relationships. Suppose that e_i and e_j lie in the same orbit of Π . Then there is a ‘path’ in the structure connecting the fiber containing e_i to the fiber containing e_j , i.e. there is $\mathbf{i} \in n^{<\omega}$ such that $\mathbf{U}_{\mathbf{i}} e_i \in \pi^{-1}(\pi(e_j))$. We wish to construct an existential sentence θ_{ij} that codes this path. Writing e_i^0 for e_i , our candidate for θ_{ij} is the following:

$$\exists \gamma_{ij} \exists_{k=1}^p b_{ijk} \exists_{k=1}^p e_i^k \left(\begin{array}{l} \bigwedge_{k=1}^p \mathbf{E}(e_i^k, \eta_{i_k} \pi(e_i^{k-1})) \wedge \bigwedge_{k=1}^p \mathbf{U}_{i_k} e_i^{k-1} = \lambda_{i_k}(b_{ijk}) e_i^k \\ \wedge P_{i_k}(b_{ijk}, \pi(e_i^{k-1})) \wedge e_j = \gamma_{ij} e_i^p \end{array} \right)$$

This sentence is, of course, satisfied in (L, k) . We now have enough information to construct a class of formulas with which to prove quantifier elimination.

Definition 4.2. Let $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ and $x = (x_1, \dots, x_r)$ be tuples of variables from the sorts L and k respectively. A **core formula** with variables (v, x) is defined to be a formula of the following shape:

$$\exists \lambda \exists_{i=1}^t e_i \exists_{i=1}^t y_i \exists \gamma \exists b \left(\begin{array}{l} \bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij}) \\ \wedge S(\lambda, y, \gamma, b, x) \end{array} \right)$$

where

- (1) Θ is a subset of $\{(i, j) : 1 \leq i, j \leq t\}$.
- (2) S defines a Zariski constructible subset of $k^{r_1} \times V^t \times \Gamma^{r_2}$ where
 - (a) $r_1 = l(x) + l(b) + s$ (l denotes length)
 - (b) $r_2 = l(\gamma)$
- (3) ϕ_{ij} is θ_{ij} with the existential quantification over γ_{ij}, b_{ik} removed.

A **core type** is defined to be a consistent collection of core formulas. If (v, a) is a tuple of elements from $L^s \times k^r$, the **core type of** (v, a) (denoted $\text{ctp}(v, a)$) is defined to be the set of all core formulas satisfied by (v, a) .

We now wish to demonstrate that the type of a tuple is determined by its core type.

Proposition 4.4. Let $(L, k) \models T_A$ be \aleph_0 -saturated. Suppose that $(v, c), (w, d)$ are both tuples from $L^s \times k^r$ with the property that $\text{ctp}(v, c) = \text{ctp}(w, d)$. Then $\text{tp}(v, c) = \text{tp}(w, d)$.

Proof. We shall construct an automorphism $\tilde{\sigma}$ of (L, k) with the property that $\tilde{\sigma} : (v, c) \mapsto (w, d)$. Re-index the tuple v as $\{v_{ij} : 1 \leq i \leq t; 1 \leq j \leq s_i, \sum_i s_i = s\}$ so that given v_{ij}, v_{kl} , we have $i = k$ if and only if $\pi(v_{ij}) = \pi(v_{kl})$. By what has already been discussed, the axioms provide us with:

- (1) Tuples a_i^1 such that $\pi(v_{ij}) = a_i^1$ for every $1 \leq i \leq t$.
- (2) Basis elements $e_i^1 \in \mathbf{E}(L, a_i)$ and scalars λ_{ij}^1 such that

$$\bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \lambda_{ij}^1 e_i^1 = v_{ij}$$

holds.

Now we construct the set Θ so that $(i, j) \in \Theta$ if and only if there is a path from $\pi^{-1}(a_i^1)$ to $\pi^{-1}(a_j^1)$; namely there is a sequence $\mathbf{i} \in n^{<\omega}$ such that $\mathbf{U}_{\mathbf{i}} e_i^1 \in \pi^{-1}(a_j^1)$. Note then that there is a corresponding existential sentence θ_{ij} that codes this path. Thus the following conjunct holds in (L, k) :

$$\bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = a_i^1 \wedge \lambda_{ij}^1 e_i^1 = v_{ij} \wedge \mathbf{E}(e_i^1, a_i^1) \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i^1, e_j^1, b_{ij}^1, \gamma_i^1)$$

We shall denote the above formula by $\phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1)$. Consider the following set of formulas:

$$\Sigma = \{ \phi(w, e', x', \lambda', \gamma', b') \wedge S(x', \lambda', \gamma', b', d) : \\ (L, k) \models \phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, c) \}$$

Here the variables have been primed to distinguish them from actual parameters. The S range over all constructible subsets of an appropriate cartesian power of k .

Claim: Σ is consistent.

Proof. We show that Σ is finitely consistent. By definition Σ is closed under finite conjunctions, so let $\phi \wedge S \in \Sigma$. Then

$$(L, k) \models \phi(v, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, c)$$

Existentially quantifying out $e^1, a^1, \lambda^1, \gamma^1$ and b^1 , we obtain a core formula satisfied by (v, c) . But $\text{ctp}(v, c) = \text{ctp}(w, d)$, so there are $e^2, a^2, \lambda^2, \gamma^2, b^2$ such that

$$(L, k) \models \phi(w, e^1, a^1, \lambda^1, \gamma^1, b^1) \wedge S(a^1, \lambda^1, \gamma^1, b^1, d)$$

as required. \square

By saturation, the type Σ is satisfied by a tuple $(e^2, a^2, \lambda^2, \gamma^2, b^2)$ say. In particular, we have that

$$\text{tp}^k(a^1, \lambda^1, \gamma^1, b^1, c) = \text{tp}^k(a^2, \lambda^2, \gamma^2, b^2, d)$$

and by saturation of k we therefore obtain an isomorphism σ of k such that

$$\sigma : (a^1, \lambda^1, \gamma^1, b^1, c) \mapsto (a^2, \lambda^2, \gamma^2, b^2, d)$$

It remains to extend σ to the whole of (L, k) . Partition V up into orbits of Π :

$$V = \prod_{x \in \Lambda} \Pi x$$

for some set of representatives Λ . Let $x \in \Lambda$ be such that Πx contains $a_{i_1} \dots a_{i_q}$ and no other a_i . By re-indexing if necessary, we can assume that the a_{i_j} are listed in order of increasing length with respect to x . We may also assume that $x = a_{i_1}^1$ by changing the set of representatives if necessary. We carry out an induction on length which is identical to the proof of Theorem 4.1. One only has to note that the construction automatically maps $e_{i_j}^1 \mapsto e_{i_j}^2$ for every $1 \leq j \leq q$. Indeed, suppose that there is a path from $\pi^{-1}(a_{i_1}^1)$ to $\pi^{-1}(a_{i_j}^1)$. Then it is coded in θ_{i_1, i_j} . But $\phi_{i_1, i_j}(e_{i_1}^1, e_{i_j}^1, b_{i_1, i_j}^1, \gamma_{i_1, i_j}^1)$ holds, hence so does $\phi_{i_1, i_j}(e_{i_1}^2, e_{i_j}^2, b_{i_1, i_j}^2, \gamma_{i_1, i_j}^2)$ by the fact that Σ is realized by $(w, e^2, a^2, \lambda^2, \gamma^2, b^2)$. Thus the following conjunctions hold in (L, k) for $l = 1, 2$:

$$\bigwedge_{k=1}^n \mathbf{E}(e_{i_1}^{lk}, \eta'_{jk} \pi(e_{i_1}^{l, k-1})) \wedge \bigwedge_{k=1}^n \mathbf{U}_{jk} e_{i_1}^{l, k-1} = \lambda_{jk}(b_{i_1, i_j, k}^l) e_{i_1}^{lk} \wedge P_{jk}(b_{i_1, i_j, k}^l, \pi(e_{i_1}^{l, k-1})) \\ \wedge e_{i_j}^l = \gamma_{i_1, i_j}^l e_{i_1}^{ln}$$

where $e_{i_1}^{l0} = e_{i_1}^l$.

Claim: $\tilde{\sigma}(e_{i_1}^{1k}) = e_{i_1}^{2k}$ for every $1 \leq k \leq n$.

Proof. This holds by fiat for $k = 0$. In constructing $\tilde{\sigma}$, we will have selected $e' \in \pi^{-1}(\eta_{j_1}(a_{i_1}^2))$ such that

$$\mathbf{U}_{j_1} e_{i_1}^2 = \lambda_{j_1}(b_{i_1, i_j, 1}^2) e' \wedge P_{j_1}(b_{i_1, i_j, 1}^2, a_{i_1}^2)$$

holds. But $\sigma(b_{i_1, i_j, 1}^1) = b_{i_1, i_j, 1}^2$, hence we have $\tilde{\sigma}(e_{i_1}^{11}) = e'$. But $\mathbf{U}_{j_1} e_{i_1}^2 = \lambda_{j_1}(b_{i_1, i_j, 1}^2) e_{i_1}^{21}$ by the long conjunct, hence $\tilde{\sigma}(e_{i_1}^{11}) = e_{i_1}^{21}$. Now we repeat the argument till we reach $k = n$. \square

Because $\sigma(\gamma_{i_1, i_j}^1) = \gamma_{i_1, i_j}^2$, it now follows that $\tilde{\sigma}(e_{i_j}^1) = e_{i_j}^2$ as required. \square

It follows by compactness that every \mathcal{L}_A -formula with parameters from k is equivalent to a boolean combination of core formulas. Some further analysis reveals the structure of subsets of (L, k) defined using parameters from both L and k . Intuitively, these should be determined by a class of formulas similar to core formulas, the only difference being that these formulas can also express information about how bases from the fibers containing these parameters from L are connected to other fibers via paths.

Definition 4.3. Let e' be a tuple of elements from L with length p such that all e'_i are basis elements. Let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_n)$ be tuples of variables from L . A **general core formula** with variables (v, w) over e' is defined to be a formula of the following shape:

$$\exists \lambda \exists \mu \exists_{i=1}^s e_i \exists_{i=1}^s y_i \exists \gamma \exists b \left(\bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge S(\lambda, \mu, y, \gamma, b, x) \right)$$

where

- (1) $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$ is an appropriate enumeration of v
- (2) $\Theta \subseteq \{(i, j) : 1 \leq i, j \leq s\}$
- (3) S is a constructible subset of $k^{r_1} \times V^s \times \Gamma^{r_2}$ where
 - (a) $r_1 = l(x) + l(b) + m + n$
 - (b) $r_2 = l(\gamma)$
- (4) ϕ is defined to be

$$\bigwedge_{i=1}^p \bigwedge_{j=1}^{p_i} \mu_{ij} e'_i = w_{ij} \wedge \bigwedge_{(i,j) \in \Theta_1} \phi_{ij}(e'_i, e_j, b_i, \gamma_{ij}) \wedge \bigwedge_{(i,j) \in \Theta_2} \phi_{ij}(e_i, e'_j, b_i, \gamma_{ij})$$

where

$$\Theta_1 \subseteq \{(i, j) : 1 \leq i \leq p, 1 \leq j \leq s\} \quad \Theta_2 \subseteq \{(i, j) : 1 \leq i \leq s, 1 \leq j \leq p\}$$

and $\{w_{ij} : 1 \leq i \leq p, 1 \leq j \leq p_i\}$ is an appropriate enumeration of w .

We shall denote such a formula by $\exists e S$ and call S the **Zariski constructible component** of $\exists e S$.

Proposition 4.5. Let $(L, k) \models T_A$. If ϕ is formula with parameters from L, k then it is equivalent to a boolean combination of general core formulas.

Proof. Suppose that $\phi(v, x)$ is a formula with free variables (v, x) over a finite set of parameters $w = (w_1, \dots, w_p)$ of L and some unspecified parameters from k . Then $\phi(v, x)$ is equivalent to some $\phi_1(v, w, x)$ where $\phi_1(v, w', x)$ is a formula with free variables (v, w', x) merely over some set of parameters from k . Hence ϕ_1 is equivalent to a boolean combination of core formulas over k by Proposition 4.4. Thus it suffices to prove that a core formula over k with free variables (v, w', x) is equivalent to a boolean combination of general core formulas after substituting w' with the tuple w . It transpires that we have the stronger result that every core formula is equivalent to a finite disjunction of general core formulas after substitution, which we now show.

So let $\varphi(v, w', x)$ be a core formula. We can fix an enumeration $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i, \sum_i s_i = n\}$ of (v, w') such that

- (1) n is the length of (v, w') .
- (2) $\pi(v_{ij}) = \pi(v_{kl})$ if and only if $i = k$.
- (3) Those v_{ij} for which v_{ij} is not in w' for any j are listed first, i.e. there is a maximum $m \leq s$ such that $v_{ij} \notin w'$ for all $i \leq m$.
- (4) For $i > m$, the w' variables are listed last, i.e. there is a minimum $t_i \leq s_i$ such that $v_{ij} \in w'$ for all $i > m$.

Now $\varphi(v, w', x)$ looks like

$$\exists \lambda \exists_{i=1}^t e_i \exists_{i=1}^t y_i \exists \gamma \exists b \left(\bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij}) \wedge S(\lambda, y, \gamma, b, x) \right)$$

for some $\Theta \subseteq \{(i, j) : 1 \leq i, j \leq s\}$ and S over k . Substitute w for w' . The resulting formula can be simplified by noting that some of the information it expresses is already contained in the theory. If $k > m$, then $y_k = \pi(w_{kl})$ is determined, thus $\exists y_k$ and such conjuncts can be dropped for $k > m$. Moreover, $\exists e_k$ can be dropped by replacing the formula with a finite disjunction, where each disjunct contains e'_k for e_k and e'_k ranges over the finitely many canonical basis elements of $\pi^{-1}(y_k)$. This allows us to make further deletions from each disjunct, namely $\mathbf{E}(e'_k, y_k)$ (which trivially holds) and $\lambda_{kl} e'_k = w_{kl}$ for $l > t_k$ (because λ_{kl} is determined), and we can therefore drop $\exists \lambda_{kl}$. This leaves us with the formula

$$\bigvee_{\substack{e' = (e_{t+1}, \dots, e_s) \\ e'_k \in \pi^{-1}(\pi(w_{kl})) \wedge \mathbf{E}(e'_k, \pi(w_{kl}))}} \exists \lambda \exists_{i=1}^m e_i \exists_{i=1}^m y_i \exists \gamma \exists b \left(\bigwedge_{i=1}^m \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \right) \wedge \mathbf{E}(e_i, y_i) \wedge \phi'$$

for appropriate ϕ' which we now determine. Clearly in

$$\bigwedge_{(i,j) \in \Theta} \phi_{ij}(e_i, e_j, b_{ij}, \gamma_{ij})$$

if we substitute the parameters e'_k for $k > m$ then some conjuncts are eliminable; namely those $\phi_{kl}(e'_k, e'_l, b_{kl}, \gamma_{kl})$ for $k, l > m$ (the theory itself tells us about paths that connect the fibers containing these e'_k, e'_l). Hence the quantifiers $\exists b_{kl}$ and $\exists \gamma_{kl}$ can also be eliminated from each disjunct. Define the sets

$$\Theta_1 = \{(i, j) \in \Theta : 1 \leq i \leq m, m < j \leq s\} \quad \Theta_2 = \{(i, j) \in \Theta : m < i \leq s, 1 \leq j \leq m\}$$

$$\Phi = \{(i, j) \in \Theta : 1 \leq i, j \leq m\}$$

Then we have ϕ' as the formula

$$\bigwedge_{i=m+1}^s \bigwedge_{j=1}^{t_i} \lambda_{ij} e'_i = v_{ij} \wedge \bigwedge_{(i,j) \in \Phi} \phi_{ij} \wedge \bigwedge_{i=1}^2 \bigwedge_{(i,j) \in \Theta_i} \phi_{ij} \wedge S'(\lambda, y, \gamma, b, x)$$

where S' is S with the determined parameters $\lambda_{kl}, y_k, b_{kl}$ and γ_{kl} substituted for the appropriate variables. Now re-label, putting $\mu_{ij} = \lambda_{i+m,j}$. We see that each disjunct is a general core formula as required. \square

4.3. Constructibility. From now on, we fix an equivariant algebra A , a Γ -rigid theory T_A and model (L, k) of T_A . Proposition 4.5 suggests taking sets of the form $\exists eC$ (where C defines a closed subset of a cartesian power of k) as giving us the closed subsets of a topology on the sorts of (L, k) and their cartesian powers. As expected, it is possible to prove that all definable subsets are constructible after taking some technicalities (adapted suitably from [Zil06]) into account. Namely, given that elements of Γ may occur as parameters in S for some general core formula $\exists eS$, we require some formalism dealing with how S transforms under applications of Γ to basis elements in the fibers.

Definition 4.4. Let $\exists eC$ be a general core formula with C giving a closed subset of $k^{r_1} \times V^s \times \Gamma^{r_2}$. We define the **action** of $\delta \in \Gamma^{r_2}$ on C to be

$$C^\delta = \{(\lambda_{ij}, \mu, y, \gamma, b, a) : (\delta_i^{-1} \lambda_{ij}, \mu, y, \delta \cdot \gamma, b, a) \in C\}$$

where

$$\delta \cdot \gamma = \begin{cases} \delta_i^{-1} \gamma_{ij} \delta_j & (i, j) \in \Theta \\ \gamma_{ij} \delta_j & (i, j) \in \Theta_1 \\ \delta_i^{-1} \gamma_{ij} & (i, j) \in \Theta_2 \end{cases}$$

C is defined to be Γ -**invariant** if $C^\delta = C$ for every $\delta \in \Gamma^{r_2}$.

The motivation for this definition is as follows. If a tuple (v, w, a) satisfies $\exists eC$, then there are $\lambda, \mu, e, y, \gamma, b$ such that

$$\bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge C(\lambda, \mu, y, \gamma, b, a)$$

holds. Put $e'_i = \delta_i e_i$ for every i . Then one can see that (v, w, a) satisfies

$$\bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda'_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge C(\lambda', \mu, y, \delta \cdot \gamma, b, a)$$

where $\lambda'_{ij} = \delta_i^{-1} \lambda_{ij}$ and $\delta \cdot \gamma$ is defined as above, only if $C^\delta(\lambda, \mu, y, \gamma, b, a)$ holds. In particular, if C is Γ -invariant then for any $\delta \in \Gamma^{r_2}$ a base change of this kind can be carried out without affecting validity.

Lemma 4.1. *Let $\exists e C_1, \exists e C_2$ be general core formulas with the same enumeration of variables. Let C_1, C_2 be Zariski closed and suppose that C_2 is Γ -invariant. Then*

- (1) $(L, k) \models \exists e(C_1 \wedge C_2) \leftrightarrow \exists e C_1 \wedge \exists e C_2$
- (2) $(L, k) \models \exists e(\neg C_2) \leftrightarrow \neg \exists e C_2$

Proof. (1) Left to right is trivial. Conversely, if the right-hand side holds for a tuple (v, w, a) , then we may obtain different basis elements e and e' as witnesses to $\exists e C_1$ and $\exists e C_2$ respectively. But the Γ -invariance of C_2 means that we can transform e' to e without affecting validity. So the left-hand side holds.

- (2) Right to left is easy. Conversely, suppose that (v, w, a) satisfies $\exists e(\neg C_2)$ and that e is a tuple of basis elements witnessing this. If some basis elements e' witness $\exists e C_2$ then we can transform e' to e , and using the Γ -invariance of C_2 we get a contradiction. \square

Lemma 4.2. *If $\exists e S$ is a general core formula then it is equivalent to a disjunction of general core formulas of the type $\exists e(C_1 \wedge \neg C_2)$ where C_1, C_2 are Zariski closed and C_2 is Γ -invariant.*

Proof. More or less the same as [Zil06]. Fix a tuple $a \in k$ and recall that $tp^k(a)$ denotes the type of a in the language of fields. Put

$$\Sigma(a) = \{C_1 \wedge \neg C_2 : (L, k) \models (C_1 \wedge \neg C_2)(a) \text{ and } C_2 \text{ is } \Gamma\text{-invariant}\}$$

Then it suffices to prove (by Propositions 4.4 and 4.5) that $\Sigma(a) \models tp^k(a)$. By quantifier-elimination for k and noting that every constructible subset is a disjunction of conjuncts of the kind $C_1 \wedge \neg C_2$, it remains to prove that C_1, C_2 (where $C_1 \wedge \neg C_2 \in tp^k(a)$) can be replaced with \tilde{C}_1, \tilde{C}_2 (respectively) such that \tilde{C}_2 is Γ -invariant and $(L, k) \models (\tilde{C}_1 \wedge \neg \tilde{C}_2) \rightarrow (C_1 \wedge \neg C_2)$. Put

$$\tilde{C}_2 = \bigvee_{\delta \in \Gamma^{r_2}} C_2^\delta$$

Evidently \tilde{C}_2 is closed, Γ -invariant and $\neg \tilde{C}_2$ implies $\neg C_2$. If $\tilde{C}_2 \in p = tp^k(a)$ then we are done. Otherwise $\neg C_2 \wedge \tilde{C}_2 \in p$. Let Δ be the maximal (hence proper) subset of Γ^{r_2} consisting of those δ such that

$$\neg D = \bigwedge_{\delta \in \Delta} \neg C_2^\delta \in p$$

Δ is non-empty because $1 \in \Delta$. Put

$$\text{Stab}(\Delta) = \{\delta \in \Gamma^{r_2} : \delta \Delta = \Delta\}$$

If $\delta \notin \text{Stab}(\Delta)$ then by maximality of Δ we have $\neg D^\delta \wedge \neg D \notin p$, hence $D^\delta \in p$. Thus

$$\bigwedge_{\delta \in \Gamma^{r_2} \setminus \text{Stab}(\Delta)} D^\delta \in p$$

Claim: We have

$$(L, k) \models \bigwedge_{\delta \in \Gamma^{r_2} \setminus \text{Stab}(\Delta)} D^\delta \wedge \bigvee_{\delta \in \Gamma^{r_2}} \neg D^\delta \rightarrow \bigvee_{\delta \in \text{Stab}(\Delta)} \neg D^\delta$$

Proof. Suppose that $b \in k$ is such that $D^\delta(b)$ holds for every $\delta \in \Gamma^{r_2} \setminus \text{Stab}(\Delta)$ and $\neg D^{\delta_1}(b)$ holds for some $\delta_1 \in \Gamma^{r_2}$. Then $\delta_1 \in \text{Stab}(\Delta)$ and the claim follows. \square

The latter disjunct is clearly equivalent to $\neg D$ and $\neg D$ implies $\neg C_2$. So we take

$$\tilde{C}_1 = C_1 \wedge \bigwedge_{\delta \in \Gamma^{r_2} \setminus \text{Stab}(\Delta)} D^\delta$$

and replace \tilde{C}_2 with $\bigwedge_{\delta \in \Gamma^{r_2}} D^\delta$. The result now follows. \square

Proposition 4.6. *All definable subsets of (L, k) are constructible, namely every definable subset of (L, k) is a boolean combination of those defined by general core formulas $\exists e C$ where C is Zariski closed and Γ -invariant.*

Proof. This is almost immediate by Proposition 4.5 and Lemmas 4.1 and 4.2. One has to note, additionally that if we have a general core formula $\exists eC$ where C is just closed, we can replace C with

$$\tilde{C} = \bigvee_{\delta \in \Gamma^{r_2}} C^\delta$$

to obtain something closed and Γ -invariant. \square

5. MODEL THEORY OF EQUIVARIANT STRUCTURES: II

We conclude our analysis of models of Γ -rigid T_A by demonstrating that they are Zariski structures. Intuitively, by inspection of general core formulas one expects all the relevant properties to be verified to reduce predictably to the corresponding properties for algebraic varieties. Vaguely speaking, the Zariski constructible components ‘dominate’ the geometry. We fix an equivariant algebra A , Γ -rigid theory T_A and model $(L, k) \models T_A$.

5.1. Topology on (L, k) . We introduce a topology on $L^n \times k^m$ by taking as a basis of closed subsets those subsets that are defined by general core formulas $\exists eC(v, w, x)$ ((v, w, x) a tuple of variables from $L^n \times k^m$) where C is Zariski closed and Γ -invariant. Closed subsets are given by finite unions and arbitrary intersections of basic closed subsets. Note that if $n = 0$, then these formulas reduce to those of the form $C(x)$ where C defines a Zariski closed subset of k^m . Thus the topology on (L, k) gives us the classical Zariski topology on the sort k and its cartesian powers.

Lemma 5.1. *Let $\exists eC_1, \exists eC_2$ be general core formulas defining basic closed subsets and suppose that both formulas have the same enumeration of v variables. Then*

$$(L, k) \models \exists eC_1 \leftrightarrow \exists eC_2 \Rightarrow (L, k) \models C_1 \leftrightarrow C_2$$

Proof. By Lemma 4.1, $\exists eC_1 \wedge \neg \exists eC_2$ is equivalent to $\exists e(C_1 \wedge \neg C_2)$ hence $C_1 \wedge \neg C_2$ must be inconsistent. The rest of the lemma follows by symmetry. \square

Although a general core formula $\exists eS$ was defined with respect to two tuples of variables $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_n)$, we shall henceforth amalgamate these into one tuple which we enumerate as $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$ where there is $t \leq s$ for which $v_{ij} \in w$ for all $i > t$.

Proposition 5.1. *The topology defined on (L, k) is Noetherian.*

Proof. Suppose for contradiction that $(\exists eC_i : i \in \mathbb{N})$ defines an infinite descending chain of basic closed subsets, i.e. we have proper inclusions $\exists eC_i(L, k) \supset \exists eC_{i+1}(L, k)$ for every i . Because there are only finitely many ways of enumerating the variables v as $\{v_{ij} : 1 \leq i \leq s, 1 \leq j \leq s_i\}$, there are infinitely many $\exists eC_i$ with the same enumeration. Hence we can assume, without loss of generality, that all $\exists eC_i$ have the same enumeration of v variables. By Lemma 4.1,

$$\exists eC_{i+1}(L, k) = (\exists eC_i \wedge \exists eC_{i+1})(L, k) = \exists e(C_i \wedge C_{i+1})(L, k)$$

By Lemma 5.1 it follows that $C_i(k) \supseteq C_{i+1}(k)$. Because $\exists eC_i(L, k) \supset \exists eC_{i+1}(L, k)$, Lemma 4.1 gives that $\exists e(C_i \wedge \neg C_{i+1})$ is satisfiable. Thus we have proper inclusions $C_i(k) \supset C_{i+1}(k)$ for every i , contradicting that the Zariski topology is Noetherian. \square

5.2. Finer Results on Projections and Intersections. We now work towards the proof that (L, k) is a Zariski structure. A quick glance at the axioms to be verified will indicate that we require more detailed results about projections and intersections of subsets defined by general core formulas. By the results of the previous chapter, it is immediate that a projection of a constructible set is constructible. For a subset defined by a general core formula, we have more.

Proposition 5.2. *Let $\exists eS$ be a general core formula with the aforementioned convention on enumeration of variables. For a fixed $1 \leq i \leq s$, let j range over a subset $J \subseteq \{1, \dots, s_i\}$. Then $\exists v_{ij} \exists eS$ is a general core formula with Zariski constructible component equivalent to one of the following:*

- (1) $\exists_{j \in J} \lambda_{ij} S$.
- (2) $\exists_{j \in J} \mu_{i-t, j} S$
- (3) $\exists \mu_{i-t, 1} \exists (i-t, j) \in \Theta_1 b_{i-t, j} \exists (i-t, j) \in \Theta_1 \gamma_{i-t, j} \exists (k, i-t) \in \Theta_2 b_{k, i-t} \exists (k, i-t) \in \Theta_2 \gamma_{i-t, k} S$

$$(4) \quad \begin{aligned} & \exists y_i \exists_{(i,k) \in \Theta} b_{ik} \exists_{(i,k) \in \Theta} \gamma_{ik} \exists_{(j,i) \in \Theta} b_{ji} \exists_{(j,i) \in \Theta} \gamma_{ji} \\ & \exists_{(j-t,i) \in \Theta_1} b_{j-t,i} \exists_{(j-t,i) \in \Theta_1} \gamma_{j-t,i} \exists_{(i,j-t) \in \Theta_2} b_{i,j-t} \exists_{(i,j-t) \in \Theta_2} \gamma_{i,j-t} S \end{aligned}$$

Proof. The proof divides into four cases:

- (1) $1 \leq i \leq t$.
- (2) $t+1 \leq i \leq s$.
- (3) $t+1 \leq i \leq s$ and $s_i = 1$.
- (4) $1 \leq i \leq t$ and $s_i = 1$.

We deal with each of these in turn.

- (1) In this case the v_{ij} do not occur in ϕ and we can eliminate the conjuncts $\lambda_{ij} e_i = v_{ij}$, thus moving the quantifiers $\exists_{j \in J} \lambda_{ij}$ to S .
- (2) In this case $\exists v_{ij} \exists e S$ is equivalent to

$$\exists \lambda \dots \exists b \left(\bigwedge_{i=1}^t \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \exists v_{ij} \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge S(\lambda, \mu, y, \gamma, b, a) \right)$$

Recall that ϕ is

$$\bigwedge_{i=t+1}^p \bigwedge_{j=1}^{p_i} \mu_{i-t,j} e'_{i-t} = v_{ij} \wedge \bigwedge_{(i-t,j) \in \Theta_1} \phi_{i-t,j}(e'_{i-t}, e_j, b_{i-t,j}, \gamma_{i-t,j}) \wedge \bigwedge_{(i,j-t) \in \Theta_2} \phi_{i-t,j}(e_i, e'_{j-t}, b_{i,j-t}, \gamma_{i,j-t})$$

Thus $\exists v_{ij} \phi$ is equivalent to ϕ' , where the latter is ϕ but with the conjuncts $\mu_{i-t,j} e'_{i-t} = v_{ij}$ removed for $j \in J$. It follows that we can move the quantifiers $\exists_{j \in J} \mu_{i-t,j}$ to S as required.

- (3) This case is similar to 2, but more is eliminable from ϕ because we can get rid of the parameter e'_i . Hence we can eliminate $\phi_{i-t,k}(e'_{i-t}, e_k, b_{i-t,k}, \gamma_{i-t,k})$ and $\phi_{k,i-t}(e_k, e'_{i-t}, b_{k,i-t}, \gamma_{k,i-t})$. The quantifiers

$$\exists \mu_{i-t,1} \exists_{(i-t,j) \in \Theta_1} b_{i-t,j} \exists_{(i-t,j) \in \Theta_1} \gamma_{i-t,j} \exists_{(k,i-t) \in \Theta_2} b_{k,i-t} \exists_{(k,i-t) \in \Theta_2} \gamma_{k,i-t}$$

can then be moved to S .

- (4) The most is eliminable in this case. We no longer require $\mathbf{E}(e_i, y_i)$ and those conjuncts ϕ_{jk} with $(j,k) \in \Theta$ and j or k equal to i . But we can also eliminate conjuncts from ϕ , namely $\phi_{j-t,i}$ for $(j-t,i) \in \Theta_1$ and $\phi_{i,j-t}$ for $(i,j-t) \in \Theta_2$. Thus we move the quantifiers

$$\begin{aligned} & \exists y_i \exists_{(i,k) \in \Theta} b_{ik} \exists_{(i,k) \in \Theta} \gamma_{ik} \exists_{(j,i) \in \Theta} b_{ji} \exists_{(j,i) \in \Theta} \gamma_{ji} \\ & \exists_{(j-t,i) \in \Theta_1} b_{j-t,i} \exists_{(j-t,i) \in \Theta_1} \gamma_{j-t,i} \exists_{(i,j-t) \in \Theta_2} b_{i,j-t} \exists_{(i,j-t) \in \Theta_2} \gamma_{i,j-t} \end{aligned}$$

to S .

□

What if two general core formulas defining basic closed subsets of $L^n \times k^m$ each have different enumerations of variables, but we wish to determine the intersection of the subsets they define? In this case, we require a common enumeration of both formulas and Lemma 4.1 should apply, providing that the resulting Zariski constructible components (after re-enumeration) are Γ -invariant.

Lemma 5.2. *Let $\exists e C_1$, $\exists e C_2$ be general core formulas defining basic closed subsets of $L^n \times k^m$. Then $\exists e C_1 \wedge \exists e C_2$ is equivalent to a general core formula $\exists e D$ where D is equivalent to*

$$\bigwedge_{p=1}^2 C_p(\lambda, \mu, y, \gamma, b, x) \wedge \bigwedge_{\alpha_p(i,1) \sim_{12} \alpha_p(j,1)} y_i = y_j \wedge \bigwedge_{(\alpha_p(i,1), \alpha_p(j,1)) \sim_{12} (\alpha_p(k,1), \alpha_p(l,1))} b_{ij} = b_{kl} \wedge \gamma_{ij} = \gamma_{kl}$$

for some equivalence relation \sim_{12} on $\{1, \dots, n\}$.

Proof. Suppose that $\exists e C_1$ and $\exists e C_2$ have enumerations

$$\{v_{ij} : 1 \leq i \leq s_1, 1 \leq j \leq s_{1i}\} \quad \{v_{ij} : 1 \leq i \leq s_2, 1 \leq j \leq s_{2i}\}$$

respectively. Linearly enumerate the v_{ij} as $v = (v_1, \dots, v_n)$. Thus we obtain bijective maps

$$\alpha_p : \{(i,j) : 1 \leq i \leq s_p, 1 \leq j \leq s_{pi}\} \rightarrow \{1, \dots, n\}$$

for $p = 1, 2$. Now introduce the equivalence relations \sim_p on $\{1, \dots, n\}$ by

$$i \sim_p j \leftrightarrow \pi_1(\alpha_p^{-1}(i)) = \pi_1(\alpha_p^{-1}(j))$$

where π_1 is the projection onto the first coordinate. Let \sim_{12} denote the symmetric closure of the composition $\sim_2 \circ \sim_1$ (hence \sim_{12} is an equivalence relation). It is easy to see that each \sim_p refines \sim_{12} . Each equivalence class $[i]_{12}$ of \sim_{12} has a canonical representative (take the smallest i in the class) and we let $\Lambda = \{k_1, \dots, k_s : k_i < k_{i+1} \text{ for all } i < t\}$ be the set of such representatives. Then one can define a map $\tau : \{1, \dots, n\} \rightarrow \{(i, j) : 1 \leq i \leq t, 1 \leq j \leq t_i\}$ such that

- (1) $\pi_1(\tau(j)) = i$ if and only if $j \sim_{12} k_i$
- (2) $\pi_2(\tau(j)) < \pi_2(\tau(j'))$ if and only if $j < j'$

where π_2 denotes the projection onto the second coordinate, and this gives us an new enumeration of v .

Claim: For $p = 1, 2$, $\exists e C_p$ is equivalent to

$$\exists \lambda \exists \mu \exists e \exists \gamma \exists b \left(\bigwedge_{i=1}^t \bigwedge_{j=1}^{t_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge C'_p(\lambda, \mu, y, \gamma, b, x) \right)$$

where C'_p is equivalent to

$$C_p(\lambda, \mu, y, \gamma, b, x) \wedge \bigwedge_{\alpha_p(i,1) \sim_{12} \alpha_p(j,1)} y_i = y_j \wedge \bigwedge_{(\alpha_p(i,1), \alpha_p(j,1)) \sim_{12} (\alpha_p(k,1), \alpha_p(l,1))} b_{ij} = b_{kl} \wedge \gamma_{ij} = \gamma_{kl}$$

Proof. First we obtain a formula that is equivalent to $\exists e C_p$ using the linear enumeration of v given by α_p . Define $C_{\alpha_p}(\lambda, \mu, y, \gamma, b, x)$ to be C_p with the variables enumerated as follows:

- (1) $\lambda_{ij} \mapsto \lambda_{\alpha_p(i,j)}$
- (2) $y_i \mapsto y_{\alpha_p(i,1)}$
- (3) $\gamma_{ij} \mapsto \gamma_{\alpha_p(i,1), \alpha_p(j,1)}$
- (4) $b_{ijk} \mapsto b_{\alpha_p(i,1), \alpha_p(j,1), k}$

Clearly $\exists e C_p$ is then equivalent to

$$\exists_{i=1}^n \lambda \exists \mu \exists_{i=1}^n e_i \exists \gamma \exists b \left(\bigwedge_{i=1}^n \pi(v_i) = y_i \wedge \lambda_i e_i = v_i \wedge \mathbf{E}(e_i, y_i) \wedge \bigwedge_{i \sim_p j} e_i = e_j \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta'} \phi_{ij} \right. \\ \left. \wedge C_{\alpha_p}(\lambda, \mu, y, \gamma, b, x) \wedge \bigwedge_{i \sim_p j} y_i = y_j \wedge \bigwedge_{(i,j) \sim_p (k,l)} \gamma_{ij} = \gamma_{kl} \wedge b_{ij} = b_{kl} \right)$$

where $(i, j) \sim_p (k, l)$ is defined to hold if and only if $i \sim_p k$ and $j \sim_p l$; and Θ' is an appropriate subset of $\{(i, j) : 1 \leq i, j \leq n\}$. Now we define $C_{\tau^{-1} \circ \alpha_p}(\lambda, \mu, y, \gamma, b, x)$ to be C_{α_p} but with the following enumeration of variables

- (1) $\lambda_i \mapsto \lambda_{\tau(i)}$
- (2) $y_i \mapsto y_{\pi_1(\tau(i))}$
- (3) $\gamma_{ij} \mapsto \gamma_{\pi_1(\tau(i)), \pi_1(\tau(j))}$
- (4) $b_{ijk} \mapsto b_{\pi_1(\tau(i)), \pi_1(\tau(j)), k}$

Because \sim_p refines \sim_{12} , $i \sim_p j$ implies that $\pi_1(\tau(i)) = \pi_1(\tau(j))$. So using the enumeration of v given by τ , we see that the above formula is equivalent to

$$\exists \lambda \exists \mu \exists e \exists \gamma \exists b \left(\bigwedge_{i=1}^t \bigwedge_{j=1}^{t_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta''} \phi_{ij} \wedge C_{\tau^{-1} \circ \alpha_p}(\lambda, \mu, y, \gamma, b, x) \right)$$

where Θ'' is an appropriate subset of $\{(i, j) : 1 \leq i, j \leq t\}$. It is easy to see that $C_{\tau^{-1} \circ \alpha_p}$ is equivalent to C_p with the exception that some of the y_i, b_{ij}, γ_{ij} become identified according to \sim_{12} , and hence the claim follows. \square

Clearly C'_p is also Γ -invariant, hence by Lemma 4.1 we obtain the required result. \square

By definition of the topology on (L, k) , it follows by Lemma 5.2 that all closed subsets are finite unions of basic closed subsets. Thus if a closed subset \mathbf{C} is irreducible, it is basic closed.

5.3. Zariski Structure. Let $\exists eC$ be a general core formula defining a basic closed subset. By Lemma 5.2, changing the enumeration of the variables can potentially affect C by introducing identifications. Hence if $\exists eC$ is equivalent to $\exists eC'$ where the latter has a different enumeration of variables, it is possible that $\dim C(k) > \dim C'(k)$. Sticking to our philosophy (and corroborative results) that the geometry on k ‘dominates’ the geometry on (L, k) , we wish to define the dimension of $\exists eC(L, k)$ to be $\dim C(k)$ for suitable C . For this purpose we take the general core formula $\exists e\hat{C}$ defining $\exists eC(L, k)$ with the finest enumeration of v variables possible; namely such that if $\exists eC'$ also defines $\exists eC(L, k)$ then $\sim_{\hat{C}}$ refines $\sim_{C'}$ (with $\sim_{\hat{C}}$ and $\sim_{C'}$ defined as in Lemma 5.2). Such an enumeration is clearly possible because there are only finitely many possible enumerations, and we shall call $\exists e\hat{C}$ the **canonical presentation** of $\exists eC$.

Definition 5.1. Let $\exists eC$ define a closed irreducible subset of $L^n \times k^m$. We define the **dimension** of $\exists eC(L, k)$ to be

$$\dim \exists eC(L, k) := \dim \hat{C}(k)$$

For $\exists eC(L, k)$ a closed subset, $\dim \exists eC(L, k) := \max\{\mathbf{C}_i\}$ where \mathbf{C}_i are the irreducible components of $\exists eC(L, k)$. If $\exists eS(L, k)$ is constructible, its dimension is defined to be the dimension of its closure.

Lemma 5.3. Let $\exists e\hat{C}(L, k)$ be closed and irreducible. Then $\hat{C} = \bigvee D^\delta$ where D defines some closed irreducible subset of $\hat{C}(k)$.

Proof. Let D be any irreducible component of \hat{C} . Then $\bigvee_{\delta \in \Gamma^{r_2}} D^\delta$ is closed and Γ -invariant. Hence $\exists e \bigvee D^\delta$ defines a basic closed subset. Because $\exists e\hat{C}(L, k)$ is irreducible, it follows that $\exists e\hat{C}(L, k) = \exists e \bigvee D^\delta(L, k)$ and by Lemma 5.1 the result follows. \square

Lemma 5.4. The notion of dimension defined in Definition 5.1 satisfies conditions 1 – 5 of the definition of a Zariski structure (Definition 1.1).

Proof. For ease of reference, we restate the conditions to be verified:

- (1) The dimension of a point is 0.
- (2) $\dim(\mathbf{P}_1 \cup \mathbf{P}_2) = \max\{\dim \mathbf{P}_1, \dim \mathbf{P}_2\}$ for all projective subsets $\mathbf{P}_1, \mathbf{P}_2$.
- (3) For \mathbf{C} closed and irreducible in \mathbf{X}^n and \mathbf{C}_1 a closed subset of \mathbf{C} , if $\mathbf{C}_1 \neq \mathbf{C}$ then $\dim \mathbf{C}_1 < \dim \mathbf{C}$.
- (4) For \mathbf{C} irreducible and closed in \mathbf{X}^n , if $\pi : \mathbf{X}^n \rightarrow \mathbf{X}^m$ is a projection then

$$\dim \mathbf{C} = \dim \pi(\mathbf{C}) + \min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C})$$

- (5) For any irreducible closed \mathbf{C} in \mathbf{X}^n and projection map $\pi : \mathbf{X}^n \rightarrow \mathbf{X}^m$, there is a subset \mathbf{V} relatively open in $\pi(\mathbf{C})$ such that

$$\min_{a \in \pi(\mathbf{C})} \dim(\pi^{-1}(a) \cap \mathbf{C}) = \dim(\pi^{-1}(v) \cap \mathbf{C})$$

for every $v \in \mathbf{V} \cap \pi(\mathbf{C})$.

The conditions on dimensions of points and dimensions of unions are trivial. Let $\exists e\hat{C}$ define an irreducible closed subset. By Lemma 5.3, $\hat{C} = \bigvee D^\delta$ for some closed irreducible $D(k) \subseteq \hat{C}(k)$. If some $\exists e\hat{C}_1$ defines a proper closed subset of $\exists e\hat{C}(L, k)$ then it has the same enumeration of variables because both are canonically presented. Thus $\hat{C}_1(k) \subset \hat{C}(k)$ by Lemmas 4.1 and 5.1. But then $\dim(\hat{C}_1 \wedge D^\delta)(k) < \dim D^\delta(k) = \dim \hat{C}(k)$ for some δ , hence $\dim \hat{C}_1(k) < \dim \hat{C}(k)$, verifying condition 3.

Now suppose that we have a projection $\pi : L^{n_1+n_2} \times k^{m_1+m_2} \rightarrow L^{n_1} \times k^{m_1}$. Then $\pi(\exists e\hat{C}(L, k))$ is defined by $\exists e\hat{C}'$ where $\hat{C}' = \exists z\hat{C}$ for some appropriate $z \in k$. Thus it remains to prove that

$$\dim \hat{C}(k) = \dim \exists z\hat{C}(k) + \min \dim(\hat{C}(a, k))$$

where $\hat{C}(a, k) = \pi^{-1}(a) \cap \hat{C}(k)$. But this is known for algebraic varieties, thus giving us 4. Condition 5 is proved similarly. \square

Theorem 5.1. (L, k) is a Zariski structure.

Proof. By Lemma 5.4, it remains to establish semi-properness of projection maps, but this is immediate by constructibility. \square

6. MORE ON EQUIVARIANT ZARISKI STRUCTURES

We conclude our investigation of equivariant algebras and their associated structures by constructing a functor $\text{nSpec} : \text{Equiv}(k)_\Gamma^{\text{op}} \rightarrow \text{Zar}$, where $\text{Equiv}(k)_\Gamma$ is defined to be the full subcategory of $\text{Equiv}(k)$ consisting of those equivariant k -algebras whose associated theories are Γ -rigid. Some additional remarks on equivariant algebras are made.

6.1. Functorial Correspondence. We choose a candidate $\text{nSpec } A = (L, \mathbb{K})$ where \mathbb{K} is a large saturated algebraically closed field. We demonstrate that the following diagram commutes:

$$\begin{array}{ccc} B^{\text{op}} & \longrightarrow & \text{Zar}^c \\ \downarrow & & \downarrow \\ \text{Equiv}(k)_\Gamma^{\text{op}} & \longrightarrow & \text{Zar} \end{array}$$

where B is taken to be the category of commutative affine Hopf k -algebras, because it is anti-equivalent to the category of affine algebraic groups (Appendix B). The reader may wish to review the contents of Appendix B (in particular Definition B.4 and Proposition B.1) before embarking on the following results.

Lemma 6.1. *Let H be a Hopf algebra, A a H -module algebra. Suppose that $\mathbf{A}_1, \mathbf{A}_2 \in A$ are eigenvectors of the action of H on A , i.e. there are characters $\chi_i : H \rightarrow k$ such that*

$$h \cdot \mathbf{A}_i = \chi_i(h) \mathbf{A}_i \quad h \in H \quad i = 1, 2$$

Then $\mathbf{A}_1 \mathbf{A}_2$ is a H -eigenvector with character $\chi(h) = \sum_{(h)} \chi_1(h') \chi_2(h'')$ for every $h \in H$.

Proof. Given $h \in H$ we have

$$h \cdot (\mathbf{A}_1 \mathbf{A}_2) = \sum_{(h)} (h' \cdot \mathbf{A}_1) (h'' \cdot \mathbf{A}_2) = \sum_{(h)} \chi_1(h') \chi_2(h'') \mathbf{A}_1 \mathbf{A}_2$$

as required. \square

Proposition 6.1. *Given a morphism $\varphi : A \rightarrow B$ in $\text{Equiv}(k)_\Gamma$, there is a morphism of Zariski structures $\text{nSpec } \varphi : \text{nSpec } B \rightarrow \text{nSpec } A$.*

Proof. Let $\text{nSpec } A = (L_A, V_A, \mathbb{K})$, $\text{nSpec } B = (L_B, V_B, \mathbb{K})$. Suppose that A is equivariant with respect to the Hopf subalgebra H and elements $\mathbf{U}_{11}, \dots, \mathbf{U}_{1n_1}$ of A . Then there is a Hopf subalgebra H' of B such that

- (1) B is equivariant with respect to H' and $\mathbf{U}_{21}, \dots, \mathbf{U}_{2n_2}$.
- (2) $\varphi(\mathbf{U}_{1i})$ is a monomial in $\mathbf{U}_{21}, \dots, \mathbf{U}_{2n_2}$ for each i .
- (3) $\varphi|_H : H \rightarrow H'$.

Without loss of generality, we can assume that the $\varphi(\mathbf{U}_{1i})$ occur amongst the $\mathbf{U}_{21}, \dots, \mathbf{U}_{2n_2}$. Indeed, if some $\varphi(\mathbf{U}_{1i})$ does not occur amongst the \mathbf{U}_{2j} for $1 \leq j \leq n_2$, we can add it to this set of generators as a primitive. By Proposition B.1, B is a H' -module algebra under the adjoint representation. By repeated application of Lemma 6.1, $\varphi(\mathbf{U}_{1i})$ is a H' -eigenvector because it is generated by the \mathbf{U}_{2j} . Recall that by the definition of an equivariant algebra (Definition 3.2), if $\Pi' = \langle \eta'_1, \dots, \eta'_{n_2} \rangle$ is the group associated to B , then for each $1 \leq j \leq n_2$ there must be a $k \leq n_2$ such that $\eta_j^{-1} = \eta_k$. Thus by adding $\varphi(\mathbf{U}_{1i})$ as a generator, we must also add another generator so that this property remains satisfied. This can be done because $\varphi(\mathbf{U}_{1i})$ is a monomial in the \mathbf{U}_{2j} .

Now we put $B' = \{\varphi(\mathbf{U}_{1i}) : 1 \leq i \leq n_1\}$. If Π is the group associated with A , we put $\phi(\Pi)$ as the subgroup of Π' generated by those η'_j for which $\mathbf{U}_{2j} \in B'$. Because $\varphi_H : H \rightarrow H'$, we have a corresponding morphism of varieties $f : V_B \rightarrow V_A$. Partition V_B into orbits of $\phi(\Pi)$:

$$V_B = \bigcup_{x \in \Lambda'} \phi(\Pi)x$$

for some set of representatives Λ' . The map $\text{nSpec } \varphi : L_B \rightarrow L_A$ is then constructed fiberwise on orbits, using an inductive procedure analogous to the proof of Theorem 4.1. Let $\phi(v, w, x)$ define a basic closed subset of $\text{nSpec } A$; thus ϕ is of the form

$$\exists \lambda \exists \mu \exists e \exists y \exists \gamma \exists b \left(\bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \pi(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \phi \wedge \bigwedge_{(i,j) \in \Theta} \phi_{ij} \wedge C(\lambda, \mu, y, z, \gamma, b, x) \right)$$

where C is Zariski closed and Γ -invariant.

Claim: Let $\pi_A : L_A \rightarrow V_A$ (and $\pi_B : L_B \rightarrow V_B$). The preimage of $\phi(\text{nSpec } A)$ is defined by

$$\exists \lambda \exists \mu \exists e \exists e'' \exists y \exists \gamma \exists b \left(\begin{array}{l} \bigwedge_{i=1}^s \bigwedge_{j=1}^{s_i} \pi_B(v_{ij}) = y_i \wedge \lambda_{ij} e_i = v_{ij} \wedge \mathbf{E}(e_i, y_i) \wedge \\ \bigwedge_{i=1}^p \bigwedge_{j=1}^{p_i} \pi_B(w_{ij}) = z_i \wedge \mu_{ij} e'_i = w_{ij} \wedge \mathbf{E}(e'_i, z'_i) \wedge \\ \bigwedge_{(i,j) \in \Theta'} \phi'_{ij} \wedge C'(\lambda, \mu, y, \gamma, b, x) \end{array} \right)$$

where

- (1) $\Theta' \subseteq \{1 \leq i, j \leq p + s\}$
- (2) ϕ'_{ij} is ϕ_{ij} with the \mathbf{U}_{1i} replaced with their images under φ .
- (3) $C'(\lambda, \mu, y, z, \gamma, b, x)$ holds if and only if
 - $C(\lambda, \mu, f(y), \gamma, b, x)$ holds in $\text{nSpec } A$.
 - $z' = (z'_i) \in f^{-1}(z)$ where $z = (\pi_A(e'_i))$.

Proof. Immediate by construction of $\text{nSpec } \varphi$. □

Because C' is closed, the formula in the claim defines a closed set in $\text{nSpec } B$, as required. □

Corollary 6.1. *There is a functor $\text{nSpec} : \text{Equiv}(k)_\Gamma^{\text{op}} \rightarrow \text{Zar}$ extending $\text{B}^{\text{op}} \rightarrow \text{Zar}^c$.*

Proof. Immediate by Proposition 6.1. We note that if H is a commutative affine Hopf k -algebra then the structure corresponding to H is a line bundle $\pi : L \rightarrow G$ where G is an affine algebraic group. Each fiber $L_g = \pi^{-1}(g)$ is the G -module given by the character $\chi_g : G \rightarrow k$ associated to $g \in G$. It is clear that this structure is definably interpretable in k . □

A certain amount of algebraic structure is recoverable from an abstract theory T which ‘looks like’ T_A . If T is formulated in the language

$$\mathcal{L} = (L, V, k, \pi, \mathbf{E}, \mathbf{U}_i, h_j, C : 1 \leq j \leq m, 1 \leq i \leq n)$$

and satisfies the axioms of Definition 3.4, let F be the free algebra over k on the generators h_j, \mathbf{U}_i . A model $(L, k) \models T$ will be a representation of $A = F/I$ where I is the annihilator of (L, k) . By Theorem 4.1, the algebra A is determined up to the cardinality of the uncountable algebraically closed field k . It need not be the case that $T = T_A$ for some equivariant k -algebra A .

Example 6.1. *Let A be the k -algebra with generators $\mathbf{U}, \mathbf{V}^{\pm 1}$ subject to the relation*

$$\mathbf{U}\mathbf{V} = q\mathbf{V}\mathbf{U}$$

where \mathbf{V} is invertible and q is a generic parameter. Then A is not equivariant (it is not even semi-equivariant). Yet there is an abstract theory T satisfying the axioms of Definition 3.4 from which A can be recovered. T is formulated in the language $\mathcal{L} = (L, V, k, \pi, \mathbf{U}, \mathbf{V}^{\pm 1}, q)$, V is the affine line,

$$\eta_{\mathbf{V}}(x) = qx \quad \eta_{\mathbf{V}^{-1}}(x) = q^{-1}x$$

$\Gamma = \{1\}$ and $\lambda_{\mathbf{V}}(y, x) = \lambda_{\mathbf{V}^{-1}}(y, x) = y - x$. Clearly T is Γ -rigid.

Thus the full subcategory of Zar consisting of large saturated models of theories which satisfy Definition 3.4 will contain Zariski structures which do not lie in the image of nSpec . An equivalence of categories via nSpec does not therefore seem possible. The following conjecture remains open.

Conjecture 6.1. *If A is a non-commutative equivariant algebra then $\text{nSpec } A$ is a non-classical Zariski structure.*

6.2. Quantized Universal Enveloping Algebras. Despite some important examples falling under the umbrella of equivariant algebras, there are many important algebras which are not equivariant. There is one such collection of algebras that the author firmly had in mind when formulating the mathematics of the previous three chapters; namely the quantized universal enveloping algebras at generic parameter.

Definition 6.1. Let \mathfrak{g} be a finite-dimensional complex semisimple Lie algebra. Let n be the rank of \mathfrak{g} , $C = (a_{ij})$ the Cartan matrix of \mathfrak{g} with respect to a choice of Cartan subalgebra \mathfrak{h} and simple roots $\alpha_1, \dots, \alpha_n$. Let $q \in k$ be generic, $q_i = q^{a_i}$. The **quantized enveloping algebra of \mathfrak{g} over k** (denoted $U_q(\mathfrak{g})$) is the k -algebra with generators $E_i, F_i, K_i^{\pm 1}$, $1 \leq i \leq n$ subject to the relations

- (1) $K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j$ $K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$.
- (2) $K_i K_j = K_j K_i$.
- (3) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$.
- (4) The quantized Serre relations:

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-l} E_j E_i^l = 0 \quad (i \neq j)$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-a_{ij}-l} F_j F_i^l = 0 \quad (i \neq j)$$

We refer the reader to Appendix A for appropriate facts and notation concerning semisimple Lie algebras. The presence of the quantized Serre relations (if non-trivial) prevent $U_q(\mathfrak{g})$ from being semi-equivariant, let alone equivariant. Nevertheless, if we discard the quantized Serre relations, we do obtain a semi-equivariant algebra $\tilde{U}_q(\mathfrak{g})$: take $H = k[K_i^{\pm 1} : 1 \leq i \leq n]$ and equip it with the group Hopf algebra structure, namely that given by

$$\Delta(K_i) = K_i \otimes K_i \quad \epsilon(K_i) = 1 \quad S(K_i) = K_i^{-1}$$

for each i . Then E_i and F_i are eigenvectors of the adjoint action of H on $\tilde{U}_q(\mathfrak{g})$ by 1 and the remaining relations satisfied by E_i, F_i in 3 are of the required form. By Proposition 3.1 there is an equivariant algebra $U'_q(\mathfrak{g})$ of which $\tilde{U}_q(\mathfrak{g})$ is an epimorphic image, for which the most likely candidate is the k -algebra subject to the relations 1, 2 and

$$(3) \quad [E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \quad 1 \leq i \leq n$$

Proposition 6.2. $U'_q(\mathfrak{g})$ is equivariant and its theory is Γ -rigid.

Proof. This is a straightforward generalization of the corresponding result for $U_q(\mathfrak{sl}_2(k))$. We shall assume for simplicity that \mathfrak{g} is simply laced (i.e. that $d_i = 1$ for all i). Define the vectors $\mathbf{a}_j = (a_{1j}, \dots, a_{nj})$ for $1 \leq j \leq n$ and put $\mathbf{A} = \mathbb{Z}\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$. Then $\Pi = q^{\mathbf{A}}$ acts on H (by multiplication) and the E_i, F_i move between fibers according to this action on the base. Define

$$\lambda_{E_i}(y) = -\lambda_{F_i}(y) = \frac{y_i + y_i^{-1}}{q - q^{-1}} \quad y = (y_1, \dots, y_n) \quad 1 \leq i \leq n$$

$$P_{E_i}(x, y) = P_{F_i}(x, y) = y^2 - x$$

Then these maps and polynomials satisfy the required conditions. As with the $U_q(\mathfrak{sl}_2(k))$ case, we have to be careful about picking the roots and for this purpose, we partition $(k^*)^n$ into cosets of $q^{\mathbf{A}}$:

$$(k^*)^n = \bigcup_{x \in \Lambda} q^{\mathbf{A}} x$$

where Λ is a set of representatives. For $x \in \Lambda$, choose any y such that $y^2 = x$ (i.e. $y = (y_1, \dots, y_n)$ with $y_i^2 = x_i$ for every i). If $z \in q^{\mathbf{A}} x$ then there is $\mathbf{a} \in \mathbf{A}$ such that $z = q^{\mathbf{a}} x$ and we choose the square root $q^{\mathbf{a}/2} y$ of z . The associated theory is trivially Γ -rigid (see the proof of Proposition 4.3). \square

It is unlikely that $\tilde{U}_q(\mathfrak{g})$ itself is equivariant, although the author has been unable to prove this. A calculation using the above $\lambda_{E_i}, \lambda_{F_i}, P_{E_i}$ demonstrates that they do not satisfy $E_i F_j - F_j E_i = 0$ for $i \neq j$. It also does not seem possible that a more sophisticated selection of functions and polynomials can rectify this without violating the corresponding relations for (7.1). Nevertheless, that there is an epimorphism $U'_q(\mathfrak{g}) \rightarrow \tilde{U}_q(\mathfrak{g})$ suggests (by a possible generalization of Corollary 6.1) that a putative geometric structure corresponding to $\tilde{U}(\mathfrak{g})$ could map to the Zariski structure $\text{nSpec } U'(\mathfrak{g})$.

6.3. Total Equivariance. We isolate a particularly nice class of equivariant algebras with the following definition.

Definition 6.2. *An equivariant k -algebra A is **totally equivariant** if any maximal commutative subalgebra has the structure of a Hopf algebra with respect to which A is equivariant.*

Example 6.2. *Let $\mathcal{O} = \mathcal{O}_q((k^\times)^2)$ be the quantum 2-torus, i.e. the k -algebra with generators $\mathbf{U}^{\pm 1}, \mathbf{V}^{\pm 1}$ subject to the relation*

$$\mathbf{U}\mathbf{V} = q\mathbf{V}\mathbf{U}$$

where q is generic. Then \mathcal{O}_q is totally equivariant.

Proof. The algebra \mathcal{O}_q is equivariant because $\mathcal{O}_q((k^\times)^n)$ is (see Subsection 3.2). Now any maximal commutative subalgebra H must be generated by some $c\mathbf{U}^p\mathbf{V}^q$ and its inverse, where $p, q \in \mathbb{Z}$ and $c \in k^\times$. Thus taking the additional generators $\mathbf{V}^{-q}\mathbf{U}^{1-p}, \mathbf{V}^{1-q}\mathbf{U}^{-p}$ and their inverses gives the whole of \mathcal{O}_q . It is easy to see that these generators are eigenvectors for H under the adjoint action, either directly or by use of Lemma 6.1. \square

A totally equivariant algebra A has many associated Zariski structures, each depending on the particular Hopf subalgebra chosen. For those maximal commutative subalgebras which are conjugated by an automorphism of A , there is a corresponding isomorphism of the associated Zariski structures by Corollary 6.1. In Example 6.2, one could consider the maximal commutative subalgebras $k[\mathbf{U}^{\pm 1}]$ and $k[\mathbf{V}^{\pm 1}]$. By total equivariance, there are two corresponding Zariski structures $\text{nSpec } \mathcal{O}_q$ and $\text{nSpec } \mathcal{O}'_q$ respectively. Let φ be the k -algebra automorphism of \mathcal{O}_q given by

$$\varphi(\mathbf{U}) = \mathbf{V} \quad \varphi(\mathbf{V}) = \mathbf{U}^{-1}$$

Then φ is an equivariant automorphism and it follows that there is a corresponding Zariski isomorphism $\text{nSpec } \mathcal{O}_\epsilon \simeq \text{nSpec } \mathcal{O}'_\epsilon$.

APPENDIX A. SEMISIMPLE LIE ALGEBRAS

We include some fundamental results on semisimple Lie algebras in this appendix primarily for the purpose of setting up some notation. We assume that the reader is familiar with the definitions of a Lie algebra and Lie algebra representation (in particular the adjoint representation). More details can be found in many sources, e.g. [Hum73].

A Lie algebra \mathfrak{g} is said to be **semisimple** if it has no non-zero abelian ideals. If \mathfrak{g} is semisimple it possesses a **Cartan subalgebra** \mathfrak{h} , namely a maximal abelian subalgebra consisting of ad-semisimple elements. We have an eigenspace decomposition \mathfrak{g} under the action of $\text{ad } \mathfrak{h}$ called the **Cartan decomposition**:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where

- (1) $\mathfrak{h} = \mathfrak{g}_0 = \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\}$
- (2) Φ consists of **roots**, namely those $\alpha : \mathfrak{h} \rightarrow k$ such that

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [x, y] = \alpha(y)y \text{ for all } y \in \mathfrak{h}\}$$

is non-zero.

Φ forms a **reduced root system** in \mathfrak{h}^* . By the properties of root systems, Φ contains a subset Φ^+ of **positive roots** and Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. There is a basis Δ of \mathfrak{h}^* consisting of **simple roots**:

$$\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi^+$$

Here $n = \dim_k \mathfrak{h}$ is called the **rank** of \mathfrak{g} . The Lie algebra \mathfrak{g} splits as a direct sum

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \quad \mathfrak{n}^\pm = \sum_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$$

called the **triangular decomposition** of \mathfrak{g} .

The **Killing form** $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ is the symmetric bilinear form defined by $(x, y) = \text{Tr}(\text{ad } x \circ \text{ad } y)$. It is \mathfrak{g} -invariant and non-degenerate. Moreover, the Killing form is non-degenerate when restricted to \mathfrak{h} . The **Cartan matrix** of \mathfrak{g} is the $n \times n$ matrix $C = (a_{ij})$ with

$$a_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

Any reduced root system is the sum of irreducible root systems. The latter correspond to **simple** Lie algebras. If \mathfrak{g} is simple, the simple roots of \mathfrak{g} fall under the following two cases:

- (1) They are all of the same length (i.e. (α, α) is the same for every simple α), and we are said to be in the **simply laced case**.
- (2) They have two lengths: **long** and **short**.

The form is normalized so that $(\alpha, \alpha) = 2$ for all short roots. With this normalization, the integers $d_i = (\alpha_i, \alpha_i)/2$ for $1 \leq i \leq n$ belong to $\{1, 2, 3\}$. Putting D as the diagonal matrix with entries d_1, \dots, d_n , the matrix DC is symmetric.

APPENDIX B. HOPF ALGEBRAS

More details concerning Hopf algebras can be found in [Kas94].

B.1. Coalgebras. Let k be a field.

Definition B.1. A **coalgebra** over k is a triple (C, Δ, ϵ) where C is a vector space, $\Delta : C \rightarrow C \otimes C$ and $\epsilon : C \rightarrow k$ are linear maps satisfying the following axioms:

- (1) The diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow 1_C \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes 1_C} & C \otimes C \otimes C \end{array}$$

commutes.

- (2) The diagram

$$\begin{array}{ccccc} k \otimes C & \xleftarrow{\epsilon \otimes 1_C} & C \otimes C & \xrightarrow{1_C \otimes \epsilon} & C \otimes k \\ & \searrow \simeq & \uparrow \Delta & \nearrow \simeq & \\ & & C & & \end{array}$$

commutes.

The maps Δ and ϵ are called the **coproduct** and **counit** of C respectively.

One obtains the Definition B.1 by writing out the definition of a k -algebra diagrammatically and reversing all of the arrows. Let (C, Δ, ϵ) be a coalgebra. If $x \in C$ then $\Delta(x) \in C \otimes C$, hence

$$\Delta(x) = \sum_i x'_i \otimes x''_i$$

for some $x'_i, x''_i \in C$. We adopt the **Sweedler notation**

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

to get rid of the subscripts.

B.2. Bialgebras and Hopf Algebras. It is possible to have vector spaces which come equipped with both an algebra and a coalgebra structure. Naturally these two structures should interact in some way, and this leads to the definition of a bialgebra.

Definition B.2. A *bialgebra* over k is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where

- (1) (H, μ, η) is an algebra.
- (2) (H, Δ, ϵ) is a coalgebra.
- (3) The maps Δ and ϵ are algebra homomorphisms.

Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. For $f, g \in \text{End}(H)$, define $f * g$ to be the composition of the maps

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H$$

The resulting map on $\text{End}(H)$ is bilinear and is called the **convolution**.

Definition B.3. Let $(h, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An endomorphism S of H is called an **antipode** for H if

$$S * 1_H = 1_H * S = \eta \circ \epsilon$$

A **Hopf algebra** is a bialgebra with an antipode.

Coordinate rings of algebraic groups are important examples of commutative Hopf algebras. If \mathfrak{g} is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ has Hopf algebra structure given by

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \epsilon(x) = 0 \quad s(x) = -x$$

for every $x \in U(\mathfrak{g})$. This is a **cocommutative** Hopf algebra; namely $\Delta \circ \tau = \Delta$ where $\tau(x \otimes y) = y \otimes x$ for all $x, y \in U(\mathfrak{g})$. Quantum groups, like the quantized universal enveloping algebras $U_q(\mathfrak{g})$, are neither commutative nor cocommutative.

B.3. Adjoint representation. Let H be a Hopf algebra. If a, x and elements of H , define

$$a \cdot x = \sum_{(a)} a' x S(a'') \quad x^a = \sum_{(a)} S(a') x a''$$

These endow H with the structure of a left- (respectively right-) module over itself and are called the **left-** (respectively **right-**) **adjoint representation** of H . These definitions generalize the adjoint action of a Lie algebra on itself and the action of a group on itself by conjugation.

Definition B.4. Let H be a k -bialgebra. An algebra A is a **H -module algebra** if

- (1) A is a H -module.
- (2) The multiplication $\mu : A \otimes A \rightarrow A$ and unit $\eta : k \rightarrow A$ of A are morphisms of H -modules.

In Definition B.4, the field k is equipped with H -module structure given by $h \cdot c = \epsilon(h)c$ for every $h \in H$ and $c \in k$. The tensor product $A \otimes A$ is equipped with the following H -module structure:

$$h \cdot (a \otimes b) = \Delta(h)(a \otimes b) = \sum_{(h)} (h' \cdot a) \otimes (h'' \cdot b)$$

Proposition B.1. The left (respectively right) adjoint representation of H turns H into a left (respectively right) H -module algebra.

Proof. We prove this for the left adjoint representation; the proof for the right adjoint representation being analogous. Firstly, for $a, x \in H$ we have

$$a \cdot \eta(1) = \sum_{(a)} a' S(a'') = \epsilon(a) 1_H$$

where 1_H is the identity element of H . For $a, x, y \in H$,

$$\begin{aligned}
 \sum_{(a)} (a' \cdot x)(a'' \cdot y) &= \sum_{(a)} a' x S(a'') a''' y S(a''') \\
 &= \sum_{(a)} a' x \epsilon(a'') y S(a''') \\
 &= \sum_{(a)} a' x y S(a'') \\
 &= a \cdot (xy)
 \end{aligned}$$

as required. \square

REFERENCES

- [AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., 1969.
- [BK04] E. Backelin and K. Kremnitzer, *Quantum Flag Varieties, Equivariant Quantum D-modules and Localization of Quantum Groups*, arXiv: math/0401108v2, 2004.
- [B99] E. Bouscaren et al, *Model Theory and Algebraic Geometry: An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture*, Lecture Notes in Mathematics, Vol. 1696, Springer, 1999.
- [BG02] K. A. Brown and K. R. Goodearl, *Lectures on Algebraic Quantum Groups*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2002.
- [CP95] V. Chari and A. N. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1995.
- [DCP93] C. De Concini, C. Procesi, *Quantum Groups, in D-modules, Representation Theory and Quantum Groups*, p. 31-140, Lecture Notes in Mathematics, Vol. 1565, 1993.
- [DI07] A. Döring and C. J. Isham, *A Topos Foundation for Theories of Physics: I -IV*, arXiv: quant-ph/0703060v1, 2007.
- [Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics Vol. 52, Springer, 1977.
- [Hru98] E. Hrushovski, *Geometric Model Theory*, Documenta Math, Proceedings ICM 1998 (Berlin), Vol. 1 (1998), pp. 281 - 302.
- [HZ96] E. Hrushovski and B. Zilber, *Zariski Geometries*, Journal of the AMS, Vol. 9, No. 1, 1996.
- [Hum73] J. E. Humphries, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics Vol. 9, Springer, 1973.
- [Hum75] J. E. Humphries, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 21, Springer, 1975.
- [Jan96] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics Vol. 6, American Mathematical Society, 1996.
- [Kas94] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Vol. 155, Springer, 1994.
- [Lang02] S. Lang, *Algebra*, Graduate Texts in Mathematics, Vol. 211, Third Edition, Springer, 2002.
- [Mah06] S. Mahanta, *On Some Approaches Towards Noncommutative Algebraic Geometry*, arXiv: math/0501166v5, 2006.
- [Mac03] A. Macintyre, *Model Theory: Geometrical and Set-Theoretic Aspects and Prospects*, Bulletin of Symbolic Logic, Vol. 9, No. 2 (Jun 2003), pp. 197 - 212.
- [MacL98] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Vol. 2, Second Edition, Springer, 1998.
- [Mar02] D. Marker, *Model Theory: An Introduction*, Graduate Texts in Mathematics, Vol. 217, Springer, 2002.
- [MR01] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, Vol. 39, American Mathematical Society, 2001.
- [Pie82] R. S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics, Vol. 88, Springer, 1982.
- [Pil96] A. Pillay, *Geometric Stability Theory*, Vol. 32 of Oxford Logic Guides, Oxford University Press, 1996.
- [Ros95] A. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Mathematics and its Applications, Vol. 330, Kluwer Academic Publishers, 1995.
- [Ros08] A. Rosenberg, *Topics in Noncommutative Algebraic Geometry, Homological Algebra and K-Theory*, MPIM Bonn preprint, 2008.
- [RTT07] R. Hotta, T. Tanisaki and K. Takeuchi, *D-Modules, Perverse Sheaves and Representation Theory*, Progress in Mathematics, Vol. 236, Birkhäuser, Boston, 2007.
- [Zil06] B. Zilber, *A Class of Quantum Zariski Geometries*, in *Model Theory with Applications to Algebra and Analysis I*, Z. Chatzidakis, H. D. Macpherson, A. Pillay, A. J. Wilkie editors, LMS Lecture Notes Series 349, Cambridge University Press, 2008.
- [Zil10] B. Zilber, *Zariski Geometries: Geometry from a Logician's Point of View*, LMS Lecture Notes Series 360, Cambridge University Press, 2010.
- [Zil11] B. Zilber, *Finitary Presheaf Associated With a Noncommutative Algebra*, unfinished draft.
- [ZS09] B. Zilber, V. Solanki, *Quantum Harmonic Oscillator as a Zariski Geometry*, arXiv: 0909.4414v1 [math.LO], 2009.